Section 6.4 Subgroups: Groups Inside a Group

**Purpose of Section**  To introduce the concept of a **subgroup** and find the subgroups of various symmetry groups.

**Introduction**

Recall the six symmetries of an equilateral triangle: the identity map, three flips about the midlines through the vertices of the triangle, and two (counterclockwise) rotations of 120 and 240 degrees.

[Symmetries of an equilateral triangle](https://example.com)

This set, along with the group operation of composition, forms a self-contained algebraic system called a group. It is distinguished by the fact the group operation is closed and the group contains an identity (do nothing operation), and every element in the group has an inverse. But this group is only the outside of the shell, inside there may be smaller groups. For example, in the case of $D_6$, the six symmetries of an equilateral triangle, consider the three symmetries, the identity map $e$ and the two rotations of 120 and 240 degrees. The Cayley table for $\{e, R_{120}, R_{240}\}$ is drawn in Figure 2, where these symmetries themselves form a group; i.e. the group operation is closed (i.e. the product of two elements belongs to the group), $e$ is the identity, and each element has an inverse.
Subgroup of rotations of symmetries of an equilateral triangle

Figure 2

This motivates the following definition of “groups within groups,” or subgroups.

**Definition:** Let \((G,\ast)\) be a group with operation \(\ast\). If a subset \(H \subseteq G\) itself forms a group with the same operation \(\ast\), then \(H\) is called a **subgroup** of \(G\). If \(H\) is neither the identity \(\{e\}\) nor the entire group \(G\), which are groups called **trivial subgroups** of \(G\), then \(H\) is called a **proper subgroup** of \(G\).

**Example 1 (Subgroups of Symmetries of an Equilateral Triangle)**

Find the proper subgroups of the dihedral group \(D_6\) the symmetries of an equilateral triangle.

**Solution:** The Cayley table for the dihedral group \(D_6\) of symmetries of an equilateral triangle and its proper subgroups are displayed in Figure 3. There are four proper subgroups of \(D_6\): the rotational subgroup \(\{e, R_{120}, R_{240}\}\) of order 3 and three “flip” subgroups \(\{e, F_v\}, \{e, F_w\}, \{e, F_m\}\), each of order 2.
### Subgroups

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>$R_{120}$</th>
<th>$R_{240}$</th>
<th>$F_v$</th>
<th>$F_{ne}$</th>
<th>$F_{nw}$</th>
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<tbody>
<tr>
<td>e</td>
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<td>$F_{nw}$</td>
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<td>$F_v$</td>
<td>$R_{240}$</td>
<td>$R_{120}$</td>
<td>$e$</td>
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</table>

$H_1 = \{ e, F_v \}$  Flip around vertical axis

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>$R_{120}$</th>
<th>$R_{240}$</th>
<th>$F_v$</th>
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<td>$e$</td>
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<td>$F_{nw}$</td>
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<td>$e$</td>
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$H_2 = \{ e, F_{nw} \}$  Flip around the northwest axis

<table>
<thead>
<tr>
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<td>$F_v$</td>
<td>$R_{240}$</td>
<td>$R_{120}$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

$H_3 = \{ e, F_{ne} \}$  Flip around the northeast axis.

$H_4 = \{ e, R_{120}, R_{240} \}$  Identity and two rotations

#### Four Proper Subgroups of Symmetries of an Equilateral Triangle

Figure 3

We let the reader verify that each of these subgroups satisfy the necessary requirements to be groups. See Problem 1.

#### Proper Subgroups of the Klein 4–Group

Recall from Section 6.1 that the group of (rotational and reflective) symmetries of a rectangle form a group, called the Klein 4–group, with elements $G = \{ e, R_{180}, H, V \}$, where as always "e" denotes the group identity, $R_{180}$ a rotation of 180 degrees, and $H, V$ flips around the horizontal and vertical midlines, respectively. Figure 4 shows the Cayley table of the symmetries of a rectangle and its three proper subgroups, all of order 2. Note how the order of the subgroups always divides the order of the group. We will not prove it here but this is a fundamental property was one of the first great theorems proven in group theory and is called Lagrange’s theorem, after the great French/Italian mathematician Joseph–Louis Lagrange (1736–1813).
Group $G = \{e, R_{180}, V, H\}$ of symmetries of a rectangle.

Subgroup of symmetries $H = \{e, H\}$ about the horizontal midline.

Subgroup of symmetries $H = \{e, V\}$ about the horizontal midline.

Subgroup of rotational symmetries $H = \{e, R_{180}\}$.

Symmetry Group of a Rectangle and Three Subgroups

Figure 4

Test of Subgroups

Although a subset $H$ of a group $G$ is a group only if it satisfies the four axioms of a group: i.e., Closure, Associativity, Identity, Inverse, the fact that $H$ is a subset of $G$, it is only necessary to verify that the group operation $\ast$ is closed in $H$ and that every element of $H$ has an inverse in $H$. There is no need to show the existence of an identity; the identity in the larger group $G$ is also an identity in the subgroup $H$. This result is summarized in the following theorem.
**Theorem 1** Let \((G,\ast)\) be a group with operation \(\ast\) and \(H\) a nonempty subset of \(G\). The set \(H\) with operation \(\ast\) is a subgroup \((H,\ast)\) of \((G,\ast)\) if the following two conditions hold:

i) \(H\) is **closed** under \(\ast\). That is, \(\forall x, y \in H \Rightarrow x \ast y \in H\).

ii) Every element in \(H\) has an **inverse** in \(H\). That is

\[
(\forall h \in H)(\exists h^{-1} \in H)(h \ast h^{-1} = h^{-1} \ast h = e).
\]

where "\(e\)" is the identity element in \(G\).

**Proof:**
Since \(\ast\) is a binary operation on \(G\) it is also a binary operation on the subset \(H\), and by assumption i) we know \(\ast\) maps \(H \times H\) into \(H\). Next, the associative law \((a \ast b) \ast c = a \ast (b \ast c)\) holds for all \(a, b, c \in H\) since \(H\) is a subset of \(G\) and we know it holds for all \(a, b, c \in G\). We now ask if the identity \(e \in H\) also belongs to \(H\) and is the identity of \(H\)? The answer is yes since by picking an \(h \in H\) we know by hypothesis ii) there exists a \(h^{-1} \in H\), and by closure \(h \ast h^{-1} = e \in H\). Hence, we have verified the four properties required for a group: **closure**, **associativity**, **identity**, and **inverse**. Hence \(H\) is a group.

**Example 2** (Test of Subgroup) Let \(G = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}\) be the group of integers with the binary operation of addition \(+\). Show the even integers \(2\mathbb{Z} = \{0, \pm 2, \pm 4, \ldots\}\) is a subgroup of \(G\).

**Solution**
We observe that \(+\) is closed binary operation in \(\mathbb{Z}\) since if \(m = 2k_1, n = 2k_2\) are even integers, so is their sum \(m + n = 2(k_1 + k_2) \in 2\mathbb{Z}\). Secondly, every even integer \(2k \in 2\mathbb{Z}\) has an inverse, namely \(-2k \in 2\mathbb{Z}\).

**Example 3** (Group of Infinite Order) Let \((G,\ast)\) be the group of points in the plane \(\mathbb{R}^2\) where the group operation \(+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is coordinate wise addition of points \((a, b) + (c, d) = (a + c, b + d)\). We leave it to the reader to show \((\mathbb{R}^2,+\)\) is a group. Show that the \(x\)-axis \(H = \{(x,0): x \in \mathbb{R}\}\) is a subgroup of \((\mathbb{R}^2,+)\).
Solution

The $x$-axis is a subset of the plane and the operation $+$ is closed in $H$ since

$$(x_1,0) \in H, (x_2,0) \in H \Rightarrow (x_1+x_2,0) \in H$$

Also every $(x_1,0) \in H$ has an inverse $(-x_1,0) \in H$, i.e. $(x_1,0) + (-x_1,0) = (0,0)$, which is the group identity in $\mathbb{R}^2$.

In general it is not a simple task to find all subgroups of a group, but for cyclic groups it is an easy task.

Subgroups of Cyclic Groups

We have seen that the finite cyclic group $Z_n$ of order $n$ is a group generated by a single element in the group. That is there exists a $g \in Z$ such that

$$\langle g \rangle = \{e, g, g^2, g^3, ..., g^{n-1}\} = Z_n.$$

To find the subgroups of $Z_n$ we start with an element $g \in Z_n$ and compute the set $\langle g \rangle$ generated by $g$. This set may or may not be all of $Z_n$, but it will be a subgroup of $Z_n$. We then move on to a new $h \in Z_n$ that is not in $\langle g \rangle$ and compute the set $\langle h \rangle$ generated by $h$. Continuing in this manner we will eventually obtain all subgroups of $Z_n$. For example to apply this technique to the group

$$Z_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$$

where the group operation is addition modulo 12, where the group operation is addition modulo 12. If we start taking “powers” of $g = 1$, we get (remember powers are really adding 1)

$$\langle 1 \rangle = \{1,2,3,4,5,6,7,8,9,10,11,0\}$$

which has generated the entire group $Z_{12}$. On the other hand the element $g = 2$ generates the subgroup $\langle 2 \rangle = \{0,2,4,6,8\} \subseteq G$. Figure 4 shows the subgroups generated by $g = 1, 2, 3, 4$. Do you see why $\langle 5 \rangle = Z_{12}$ and $\langle 6 \rangle = \{0,6\}$. 

Four typical subgroups generated by elements of the group

Figure 5

Table 1 shows the subgroups generated by each element of the group and the order of the subgroup generated by the generator.
### Table 1

<table>
<thead>
<tr>
<th>Generator</th>
<th>Order of the Generator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle 1 \rangle = \mathbb{Z}_{12} )</td>
<td>12 ( (1^{12} = 0) )</td>
</tr>
<tr>
<td>( \langle 2 \rangle = {0, 2, 4, 6, 8, 10} )</td>
<td>6 ( (2^6 = 0) )</td>
</tr>
<tr>
<td>( \langle 3 \rangle = {0, 3, 6, 9} )</td>
<td>4 ( (3^4 = 0) )</td>
</tr>
<tr>
<td>( \langle 4 \rangle = {0, 4, 8} )</td>
<td>3 ( (4^3 = 0) )</td>
</tr>
<tr>
<td>( \langle 5 \rangle = \mathbb{Z}_{12} )</td>
<td>12 ( (5^{12} = 0) )</td>
</tr>
<tr>
<td>( \langle 6 \rangle = {0, 6} )</td>
<td>2 ( (6^2 = 0) )</td>
</tr>
<tr>
<td>( \langle 7 \rangle = \mathbb{Z}_{12} )</td>
<td>12 ( (7^{12} = 0) )</td>
</tr>
<tr>
<td>( \langle 8 \rangle = {0, 4, 8} )</td>
<td>3 ( (8^3 = 0) )</td>
</tr>
<tr>
<td>( \langle 9 \rangle = {0, 3, 6, 9} )</td>
<td>4 ( (9^4 = 0) )</td>
</tr>
<tr>
<td>( \langle 10 \rangle = {0, 2, 4, 6, 8, 10} )</td>
<td>6 ( (10^6 = 0) )</td>
</tr>
<tr>
<td>( \langle 11 \rangle = \mathbb{Z}_{12} )</td>
<td>12 ( (11^{12} = 0) )</td>
</tr>
</tbody>
</table>

Generators of Subsets of \( \mathbb{Z}_{12} \)

### Note
You may have noticed that the order of the subgroups seems to always divide the order of the group. This is not a coincidence. The order of a subgroup always divides the order of a group. For example a group of order 11 will only have the trivial subgroups of the group itself and the identity subgroup. On the other hand the groups of order 6 we have seen (cyclic group of order six and the dihedral group \( D_6 \) of symmetries of an equilateral triangle both have subgroups of order 2 and 3.

**Example 4 (Subgroup Generated by \( R_{120} \))**

Find the subgroup of the dihedral group \( D_6 \) of symmetries of an equilateral triangle generated by \( R_{120} \).

**Solution**

Starting with \( \{e, R_{120}\} \) we compute \( R_{120}^2 = R_{240} \). Since this is not in \( \{e, R_{120}\} \) we include it, getting \( \{e, R_{120}, R_{240}\} \). We now compute the next power \( R_{120}^3 = e \) in which case we stop, getting the subgroup \( \langle R_{120} \rangle = \{e, R_{120}, R_{240}\} \) of rotations of \( D_3 \).
Example 5 (Subgroups of a Cyclic Group) Find the subgroups of $\mathbb{Z}_{30}$

Solution

Systematically trying different generators, we find the 8 subgroups.

\[
\langle g \rangle = \{e, g, g^2, \ldots, g^{29}\} \quad \text{(order 30)}
\]
\[
\langle g^2 \rangle = \{e, g^2, g^4, \ldots, g^{28}\} \quad \text{(order 15)}
\]
\[
\langle g^3 \rangle = \{e, g^3, g^6, \ldots, g^{27}\} \quad \text{(order 10)}
\]
\[
\langle g^5 \rangle = \{e, g^5, g^{10}, g^{15}, g^{20}, g^{25}\} \quad \text{(order 6)}
\]
\[
\langle g^6 \rangle = \{e, g^6, g^{12}, g^{18}, g^{24}\} \quad \text{(order 5)}
\]
\[
\langle g^{10} \rangle = \{e, g^{10}, g^{20}\} \quad \text{(order 3)}
\]
\[
\langle g^{15} \rangle = \{e, g^{15}\} \quad \text{(order 2)}
\]
\[
\langle g^{30} \rangle = \{e\} \quad \text{(order 1)}
\]
Problems

1. (Determination of Subgroups) Determine if the following are subgroups of $S_3$.
   a) $(1)(345)(234)$
   b) $(1)(154)(145)$

2. Show that the group defined by the following Cayley table is a subgroup of $S_3$.

   \[
   \begin{array}{c|ccc}
   * & ( ) & (123) & (132) \\
   \hline
   ( ) & ( ) & (123) & (132) \\
   (123) & (123) & (132) & ( ) \\
   (132) & (132) & ( ) & (123) \\
   \end{array}
   \]

3. For the group $G = \{e, R_{180}, H, V\}$ of symmetries of a rectangle, find $\langle R_{180} \rangle$, $\langle V \rangle$, $\langle H \rangle$. What is the order of each generator?

4. For the dihedral group $D_6 = \{e, R_{120}, R_{240}, F_v, F_m, F_n\}$ of symmetries of an equilateral triangle, find the subgroups generated by each element in the group. What is the order of each generator?

5. (\(\mathbb{R}^2\) with Addition is a Group) Let $(G, \ast)$ be the group of points in the plane $\mathbb{R}^2$ where the group operation $+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is coordinate wise addition of points $(a, b) + (c, d) = (a + c, b + d)$. Show $(\mathbb{R}^2, +)$ is a group.

6. (Generated Groups of $D_3$) In the dihedral group $D_6$ of symmetries of an equilateral triangle, find the subgroups generated by each element in the group.

7. (Generated Groups of Symmetries of a Rectangle) In the group $\{e, R_{180}, V, H\}$ of symmetries of a rectangle, find the subgroups generated by each element in the group.
8. **(Center of a Group)** The center $Z(G)$ of a group $G$ consists of all elements of the group that commute with all elements of the group. That is

$$Z(G) = \{ g \in G : gx = xg \text{ for all } x \in G \}$$

It can be shown that the center of any group is a subgroup of the group. Find the center of the group of symmetries of a rectangle. Note: The center of a group is never empty since the identity element of a group always commutes with every element of the group. The question is, are there other elements that commute with every element of the group.

9. **(Subgroup of Functions)** Let $G$ be a group of functions $f : \mathbb{R} \to \mathbb{R} - \{0\}$ with the group operation of multiplication of functions. Show that $H = \{ f \in G : f(1) = 1 \}$ is a subgroup of $G$.

10. **(Subgroup in the Complex Plane)** Let $G$ be the nonzero complex numbers with group operation multiplication. Show that $H = \{ a + bi : a, b \in \mathbb{R}, a^2 + b^2 = 1 \}$ is a subgroup of $G$. Describe the elements of $H$.

11. **(Subgroup of the Symmetry Group $S_4$)** The group of permutations of the four elements $\{1, 2, 3, 4\}$ is the symmetry group $S_4$, which has 24 elements. Show that the following permutations form a subgroup of $S_4$.

$$
e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ \end{pmatrix}, \quad (12)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ \end{pmatrix}.$$