Section 6.3 Groups of Permutations: The Symmetric Group

**Purpose of Section:** To introduce the idea of a permutation and show how the set of all permutations of a set of $n$ elements, equipped with the composition of permutations as an operation, form a group, called the symmetric group $S_n$ on $n$ elements.

Permutations and Their Products

In Section 2.3 we introduced the concept of a permutation (or arrangement) of a set of objects. We now return to the subject, but now the focus is different, instead of thinking of a permutation as an arrangement of objects (which it is of course), we think of a permutation as a one-to-one function (bijection) from a set onto itself. For example, a permutation of elements of the set $\{1, 2, 3, \ldots, n\}$ is thought of a one-to-one mapping of this set onto itself, which we represent by

$$P = \begin{pmatrix} 1 & 2 & \cdots & k & \cdots & n \\ 1^p & 2^p & \cdots & k^p & \cdots & n^p \end{pmatrix}$$

which gives the image $k^p$ of each element $k \in A$ in the first row as the element directly below in the second row.

A good way to think about permutations is the following. Consider the set of four elements $A = \{1, 2, 3, 4\}$ whose elements we permute with the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Carrying out the “shuffles” described by $P$, what will be the new arrangement of the numbers $\{1, 2, 3, 4\}$? Many beginning students do not interpret this permutation correctly, so we give you a simple explanation. A good way to think about this permutation is to think of four boxes labeled “1”, “2”, “3”, and “4” where initially inside each box contains a marble labeled with the same number; that is, box 1 contains a ball labeled 1, box 2 contains a marble labeled 2, and so on. The permutation $P$ “shuffles” the marbles as shown in Figure 1. That is, the marble in box 1 moves to box 2, the marble in box 2 moves to box 3, the marble in box 3 moves to box 4, and the marble in box 4 moves to box 1. The boxes stay fixed, the marbles inside the boxes move
according to the permutation, the net result being the permutation moves members of $A$ according to

$$P : (1,2,3,4) \rightarrow (4,1,2,3)$$

not $(2,3,4,1)$.

Another way to interpret this permutation is with a directed graph as drawn in Figure 2. Four people are standing, say from top to bottom, on four stair steps of a stairs, where the permutation results in a “rotational” movement, the top three people moving down one step, and the person at the bottom moving all the way to the top step.
Product of Permutations

We now introduce a second permutation

\[ Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \]

and carry out permutation \( P \) followed by permutation \( Q \). In other words, the composition of two functions: permutation \( P \) followed by permutation \( Q \), which gives a “reshuffling of a reshuffling” which defines the product of two permutations.

**Definition:** The composition of two permutations of a set, \( P \) followed by \( Q \) is defined as the (permutation) **product** of \( P \) and \( Q \), denoted\(^1\) by \( PQ \).

In this example, the product of the compositions is

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\(^1\) In compositions of functions \( (f \circ g)(x) = f(g(x)) \) we evaluate from “right to left”, evaluating the function \( g \) first and \( f \) second. Here, in the case of permutation functions, we have decided to evaluate from “left to right” to keep things in the spirit of “products” of members of a group which one generally things of “multiplying from left to right.”
\[ PQ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}. \]

So now, how do the four marbles in the four boxes end up after two shuffles? Figure 3 illustrates the movement of the marbles in the boxes.

Product (composition) of Two Permutations

Figure 3

A second visualization of this product is shown in Figure 4. The four marbles end up in order 1,4,3,2. (Don’t confuse boxes with marbles, the marbles move, the boxes stay fixed. The numbers in the permutations refer to the boxes, not the marbles.)
Another Representation of the Product of Permutations

Figure 4

Example 1  Find the product $PQ$ where

\[ P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \]

Solution:  Figure 5 illustrates this product.  Note $P:1 \rightarrow 4$ followed by $Q:4 \rightarrow 3$, the net result being $PQ:1 \rightarrow 3$.  In other words, we have

\[ PQ(1) = 3, \quad PQ(2) = 4, \quad PQ(3) = 1, \quad PQ(4) = 2 \]

or

\[ PQ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \]
The graph illustration of the product is shown in Figure 6.

![Composition of Two Permutations](image)

**Inverses of Permutations**

If a permutation $P$ maps $k$ into $k^P$, then the **inverse permutation** $P^{-1}$ maps $k^P$ back into $k$. In other words, the inverse of a permutation can be found by simply interchanging the top and bottom rows of the permutation $P$ and (for convenience in reading) reordering the top row in numerical order $1, 2, \ldots n$. For example

\[
Q = \begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1
\end{pmatrix} \Rightarrow Q^{-1} = \begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1
\end{pmatrix}
\]

\[
P = \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3
\end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3
\end{pmatrix}
\]

The reader can verify that

\[
PP^{-1} = QQ^{-1} = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{pmatrix}.
\]

**Cycle Notation for Permutations**

A more streamlined way to display permutations is by the use of **cycle** (or **cyclic**) notation. To illustrate how this works, consider the permutation $^2$

\[\text{Sometimes only the bottom row of the permutation is given since the first row is ambiguous. Hence, the permutation listed here could be expressed as } \{325614\}.\]
Let's now see if we can go backwards from cycle notation to recover the original form of the permutation. For example, consider

\[(14)(23) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\]

We start with the left-most cycle, where we see that 1 maps into 4 and 4 maps back into 1. This will fill in two columns of \(P\). If the first cycle does not exhaust the elements of the set, where in this example we still have the cycle (23), we continue the same process and then continue until all cycles have been used. This process will reconstruct the permutation \(P\) from its cycle notation, except we must know if any 1-cycles were dropped in the cycle notation.

**Margin Note** If you wanted to dial the telephone number 413-2567 but accidentally dialed 314-5267, then you permuted the digits according to \((25)(34)\).

**Example 2**

The following permutations are displayed both in function and cycle notation. Make sure you can go “both ways” in these equations.
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\[a) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix} = (12456)(3)\]
\[b) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)\]
\[c) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} = (1)(2)(345) = (345)\]
\[d) \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)\]
\[e) \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3) = ( )\]

Note the identity permutation in Example 1.e is sometimes written ( ).

**Margin Note:** The cycle notation was introduced by the French mathematician Cauchy in 1815. The notation has the advantage that many properties of permutations can be seen from an glance.

**Example 3 (Product of Permutations in Cycle Form)**

Find the product \(PQ\) if \(P = (125)(34)\) \(Q = (13)(45)\).

**Solution**

Applying (in succession) the permutation \(P\) first, then \(Q\) second, we see that 1 gets mapped into 2 by \(P\), then into itself by \(Q\), and hence the composition maps 1 into 2. Next, the number 2 gets mapped into 5 by \(P\) and then into 4 by \(Q\), and so the composition maps 2 into 4, and so on. Carrying out this process, we arrive at the composition in cycle form

\[PQ = (124)(35) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} \]

**Transpositions**

A permutation that interchanges two elements of a set and leaves all others unchanged is called a transposition. For example
are all transpositions. What may not be obvious is that any permutation is the product of transpositions. In other words, any permutation of elements of a set can be carried out by repeated interchanges of two elements. For example, Figure 7 shows Donald Duck, Minnie Mouse, Mickey Mouse, and Daisy Duck lined up from left to right waiting to get their picture taken. The photographer asks the three on the left to move one place to their right, and Daisy Duck to move to the left position, which is a result of the following permutation.

\[
P = \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1
\end{pmatrix} = (1234)
\]

Rotation Permutation

Figure 7

The question then arises, is it possible to carry out this maneuver by repeated interchanges of members two at a time? The answer is yes, and the answer is

\[(1234) = (12)(13)(14)\]

To see how this works, watch how they move.
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(12)

(13)

(14)
Example 4: The following permutations are written as the product of transpositions, but not necessarily in the same way. The reader can check these out.

\[(1234\ldots n) = (12)(13)(14)\ldots(ln)\]
\[(4321) = (43)(42)(41)\]
\[(15324) = (15)(13)(12)(14)\]

Symmetric Group \(S_n\)

We now see that the set of \(n!\) permutations of a set of \(n\) elements, where the product of two permutations is taken as their compositions, is a group, called the symmetric group \(S_n\).

**Theorem 1** If \(A\) is a set of \(n\) elements, then the set of all permutations of the set is a group, where the group product of two permutations \(P\) and \(Q\) is defined as the composition of \(P\) followed by \(Q\), and denoted \(PQ\). The group is called the symmetric group \(S_n\) on \(n\) elements, and the order of the group is \(|S_n| = n!|\).

**Proof:** Group multiplication is closed since each permutation (or shuffling) is a one-to-one mapping from \(A = \{1, 2, \ldots, n\}\) onto itself, so repeated permutations \(PQ\) is also a one-to-one mapping of \(\{1, 2, \ldots, n\}\) onto itself. The identity of the group is the identity mapping, i.e. the permutation that doesn’t change anything. Also, every permutation has a unique inverse since permutations are one-to-one mappings from \(\{1, 2, \ldots, n\}\) onto itself. Also, multiplication is associative since the composition of two functions is associative. Hence, the axioms of a group are satisfied.

Symmetric Group \(S_3\)

In Section 6.2 we constructed the group of rotational and reflective symmetries of an equilateral triangle, called the dihedral group \(D_3\). What we didn’t realize at the time was that this dihedral group can also be interpreted as the symmetric group \(S_3\) of all permutations of the three vertices \(\{A, B, C\}\) of the triangle. Figure 5 shows the relation between the symmetries of an equilateral triangle and the permutations of the vertices. Note that the

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3 Normally, the composition of two functions, \(P\) followed by \(Q\), is denoted \(Q \circ P\) (read right to left), but since we are focusing on group “products” we write the composition in product form \(PQ\).
composition (i.e. multiplication) of permutations acts exactly like the composition of symmetries of an equilateral triangle. When the elements of two groups can be placed in a one-to-one correspondence where the multiplication in one group is analogous to the multiplication in the other group, the groups are called “abstractly equal” or \textit{isomorphic}.

<table>
<thead>
<tr>
<th>Group of Permutations of ${A,B,C}$</th>
<th>Group of Symmetries of an Equilateral Triangle</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 = \begin{pmatrix} A &amp; B &amp; C \ A &amp; B &amp; C \ (A)(B)(C) \end{pmatrix}$</td>
<td>$\begin{tikzpicture} \node[regular triangle,draw] (A) at (0,0) {A}; \node[regular triangle,draw] (B) at (1,0) {B}; \node[regular triangle,draw] (C) at (0.5,0.866) {C}; \end{tikzpicture}$</td>
<td>Do nothing</td>
</tr>
<tr>
<td>$P_2 = \begin{pmatrix} A &amp; B &amp; C \ B &amp; C &amp; A \ (ABC) \end{pmatrix}$</td>
<td>$\begin{tikzpicture} \node[regular triangle,draw] (A) at (0,0) {A}; \node[regular triangle,draw] (B) at (1,0) {B}; \node[regular triangle,draw] (C) at (0.5,0.866) {C}; \end{tikzpicture}$</td>
<td>Counterclockwise rotation of $120^\circ$</td>
</tr>
<tr>
<td>$P_3 = \begin{pmatrix} A &amp; B &amp; C \ C &amp; A &amp; B \ (ACB) \end{pmatrix}$</td>
<td>$\begin{tikzpicture} \node[regular triangle,draw] (A) at (0,0) {A}; \node[regular triangle,draw] (B) at (1,0) {B}; \node[regular triangle,draw] (C) at (0.5,0.866) {C}; \end{tikzpicture}$</td>
<td>Counterclockwise rotation of $240^\circ$</td>
</tr>
<tr>
<td>$P_4 = \begin{pmatrix} A &amp; B &amp; C \ A &amp; C &amp; B \ (A)(BC) \end{pmatrix}$</td>
<td>$\begin{tikzpicture} \node[regular triangle,draw] (A) at (0,0) {A}; \node[regular triangle,draw] (B) at (1,0) {B}; \node[regular triangle,draw] (C) at (0.5,0.866) {C}; \end{tikzpicture}$</td>
<td>Flip through vertex $A$</td>
</tr>
<tr>
<td>$P_5 = \begin{pmatrix} A &amp; B &amp; C \ C &amp; B &amp; A \ (AC)(B) \end{pmatrix}$</td>
<td>$\begin{tikzpicture} \node[regular triangle,draw] (A) at (0,0) {A}; \node[regular triangle,draw] (B) at (1,0) {B}; \node[regular triangle,draw] (C) at (0.5,0.866) {C}; \end{tikzpicture}$</td>
<td>Flip through vertex $B$</td>
</tr>
<tr>
<td>$P_6 = \begin{pmatrix} A &amp; B &amp; C \ B &amp; A &amp; C \ (AB)(C) \end{pmatrix}$</td>
<td>$\begin{tikzpicture} \node[regular triangle,draw] (A) at (0,0) {A}; \node[regular triangle,draw] (B) at (1,0) {B}; \node[regular triangle,draw] (C) at (0.5,0.866) {C}; \end{tikzpicture}$</td>
<td>Flip through vertex $C$</td>
</tr>
</tbody>
</table>

Abstract Equivalence of $S_3$ and $D_4$

Figure 5
Cayley Table for \( S_3 \).

The six permutations of a set of three elements \( A = \{1, 2, 3\} \), written in cycle notation are listed in Table 1.

<table>
<thead>
<tr>
<th>Permutation ( P )</th>
<th>Cyclic Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P:123 \to 123 )</td>
<td>( e = ( ) )</td>
</tr>
<tr>
<td>( P:123 \to 132 )</td>
<td>( (23) )</td>
</tr>
<tr>
<td>( P:123 \to 213 )</td>
<td>( (12) )</td>
</tr>
<tr>
<td>( P:123 \to 231 )</td>
<td>( (123) )</td>
</tr>
<tr>
<td>( P:123 \to 312 )</td>
<td>( (132) )</td>
</tr>
<tr>
<td>( P:123 \to 321 )</td>
<td>( (13) )</td>
</tr>
</tbody>
</table>

Table 1

Elements of \( S_3 \)

The product \( PQ \) is the composition of \( P \) followed by \( Q \). For example the product \( PQ = (23)(12) \) is found by performing \( P = (23) \) first and \( Q = (12) \) second. Since \( P:1 \to 1 \) and \( Q:1 \to 2 \) and so \( PQ:1 \to 2 \). Also \( P:2 \to 3 \) followed by \( Q:3 \to 3 \) and so \( PQ:2 \to 3 \). Finally \( P:3 \to 2 \) and \( Q:2 \to 1 \) and so \( PQ:3 \to 1 \). In other words \( PQ = (23)(12) = (123) \) as illustrated Table 2, where as customary we suppress the writing of single cycles.

<table>
<thead>
<tr>
<th>( PQ )</th>
<th>( e = ( ) )</th>
<th>( (123) )</th>
<th>( (132) )</th>
<th>( (12) )</th>
<th>( (13) )</th>
<th>( (23) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e = ( ) )</td>
<td></td>
<td>( e )</td>
<td>( (123) )</td>
<td>( (132) )</td>
<td>( (12) )</td>
<td>( (13) )</td>
</tr>
<tr>
<td>( (123) )</td>
<td>( (123) )</td>
<td>( (132) )</td>
<td>( e )</td>
<td>( (23) )</td>
<td>( (12) )</td>
<td>( (13) )</td>
</tr>
<tr>
<td>( (132) )</td>
<td>( (132) )</td>
<td>( e )</td>
<td>( (123) )</td>
<td>( (13) )</td>
<td>( (23) )</td>
<td>( (12) )</td>
</tr>
<tr>
<td>( (12) )</td>
<td>( (12) )</td>
<td>( (13) )</td>
<td>( (23) )</td>
<td>( e )</td>
<td>( (123) )</td>
<td>( (132) )</td>
</tr>
<tr>
<td>( (13) )</td>
<td>( (13) )</td>
<td>( (23) )</td>
<td>( (12) )</td>
<td>( (132) )</td>
<td>( e )</td>
<td>( (123) )</td>
</tr>
<tr>
<td>( (23) )</td>
<td>( (23) )</td>
<td>( (12) )</td>
<td>( (13) )</td>
<td>( (123) )</td>
<td>( (132) )</td>
<td>( e )</td>
</tr>
</tbody>
</table>

Table 2

Symmetric Group \( S_3 \)
Problems

1. Given the permutations

\[ P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \]

find:

a) \( PQ \)

b) \( P^{-1} \)

c) \( QP^{-1} \)

d) \( P^2 = PP \)

e) \( (PQ)^{-1} \)

2. For permutations

\[ P = \begin{pmatrix} a & b & c & d \\ A & B & C & D \end{pmatrix}, \quad Q = \begin{pmatrix} e & f & g & h \\ E & F & G & H \end{pmatrix} \]

prove or disprove \( (PQ)^{-1} = Q^{-1}P^{-1} \).

3. Find the permutation

\[ P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ ? & ? & ? & ? & ? \end{pmatrix} \]

represented by the following cyclic products

a) \( (13)(13) \)

b) \( (123)(45)(125)(45) \)

c) \( (1432) \)

d) \( (1)(2)(53)(4) \)

e) \( (135)(42) \)

4. (Composition of Permutations) For the following permutations

\[ P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix} \]
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a) Show that \( PQ \neq QP \)
b) Verify \( (PQ)R = P(QR) \)
c) Verify \( (PQ)^{-1} = Q^{-1}P^{-1} \)

5. **Subgroup of \( S_3 \)** For the group \( S_3 \) of permutations of a set of three elements drawn in Table 2, select the subset \( \{e, (123), (132)\} \) and show that this set with the same product rule also forms a group, called a subgroup of \( S_3 \). Using the interpretation that the permutations in \( S_3 \) also represents the symmetries of an equilateral triangle, what is the interpretation of this subgroup? Are there any other subgroups of \( S_3 \)?

6. **Cycles as the Product of 2-cycles** A two-cycle is an exchange of two elements of a set, such as the permutation \( (23) \) of interchanging 2 and 3, leaving the other elements of the set unchanged. Every permutation of a finite set can be written (not uniquely) as the product of 2-cycles. Write the permutation \( (12345) \) as the product or composition of 2-cycles.

7. **Symmetric Group \( S_2 \)**

Given the set \( A = \{1, 2\} \).

a) Construct the Cayley table for the group of permutations on \( A \).
b) What is the order of this group?
c) Is the group Abelian?
d) What is the inverse of each element of the group?

8. **Transpositions** Verify the products

\[
\begin{align*}
a) \quad (1234\cdots n) &= (12)(13)(14)\cdots(1n) \\
b) \quad (214) &= (21)(24) = (24)(12) \\
c) \quad (4321) &= (43)(42)(41) \\
d) \quad (15324) &= (15)(13)(12)(14)
\end{align*}
\]

9. **Even and Odd Transpositions** In any symmetric group, the permutations can be “factored” as into an even or odd number of transitions. If the number of transitions is even, the permutation is called an **even** permutation, if the number of transpositions is odd the permutation is called **odd**. The symmetric group \( S_3 \) has six elements. There are three even and three odd permutations,
Find them. Hint: The identity permutation has 0 transitions, hence it is called an even permutation.

10. **(Subgroups of $S_3$)** The dihedral group $D_3$ of symmetries of an equilateral triangle, which is the same as the symmetric group $S_3$ of permutations of 3 objects, is displayed in the Cayley table in Figure 6. There are four subgroups of order 2 in this group, and one subgroup of order 3. Can you find them? Hint: We have denoted counterclockwise rotation of 120 degrees by $r$, hence $r^2$ is 240 degree rotation, and flips through the three vertices by $A, B, C$.

\[
e = ( ) \quad r \quad r^2 \quad A \quad B \quad C
\]

\[
e = ( )
\]

\[
r \quad r \quad r^2 \quad e \quad C \quad A \quad B
\]

\[
r^2 \quad r^2 \quad e \quad r \quad B \quad C \quad A
\]

\[
A \quad A \quad B \quad C \quad e \quad r \quad r^2
\]

\[
B \quad B \quad C \quad A \quad r^2 \quad e \quad r
\]

\[
C \quad C \quad A \quad B \quad r \quad r^2 \quad e
\]

Figure 6

Six Symmetries of an Equilateral Triangle