Section 5.3  Open and Closed Sets

Purpose of Section  To introduce metrical concepts of the real number system, such as open and closed sets, accumulation points, interior and boundary points of a set. These concepts will act as background for the Bolzano-Weierstrass and Heine-Borel theorems which following in the next section.

Introduction

Many of the central ideas of analysis¹, such as limits, series, sequences, and so on, are based on the concept of “closeness.” We have seen that the distance between two real numbers $x$ and $y$ is given by the absolute value of their difference: $|x - y|$, thus points within a distance $\delta$ from a point $x_0$ are points in the set

$$\{y : |x - y| < \delta\}$$

Also, we recall from calculus the definition of continuity of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ at a point $x = a$, which we state in the language of predicate logic:

$$\forall \epsilon > 0 \exists \delta > 0 \left[ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right].$$

Intuitively, this means that whenever $x$ is near $a$ (within a distance $\delta$), then $f(x)$ is close to $f(a)$ (within a distance $\epsilon$).

But there is a more general way of studying distance and that is by using neighborhoods and open sets, which brings us to the following definition.

¹ On the most basic level (real) analysis is the branch of mathematics that deals with the real numbers and functions of real numbers. But that characterization is only the tip of the iceberg since real analysis includes such topics as measure theory, harmonic analysis, optimization, differential equations, and many more.
**Definition:** Let $a \in \mathbb{R}$ and $\delta > 0$. A $\delta$-neighborhood of $a$ is the open interval $N_\delta(a) = (a-\delta, a+\delta)$ of radius $\delta$ centered at $a$. Alternate ways of writing this are

$$N_\delta(a) = \{x \in \mathbb{R} : a-\delta < x < a+\delta\} = \{x \in \mathbb{R} : |x-a| < \delta\}.$$

Using the concept of a neighborhood, we can restate the definition of continuity in a more general form:

$$\forall \varepsilon > 0 \exists \delta > 0 \left[ x \in N_\delta(a) \Rightarrow f(x) \in N_\varepsilon(f(a)) \right]$$

This brings us to the unifying concept of this section, and the study of “closeness”, the open set.

**Definition:** A subset of real numbers $A \subseteq \mathbb{R}$ is an open set if for every $a \in A$ there exists a $\delta > 0$ such that $N_\delta(a) \subseteq A$. That is

$$A \text{ open } \iff (\forall a \in A) (\exists \delta > 0) (N_\delta(a) \subseteq A)$$

In the definition of an open set, when we say “there exists a $\delta$ greater than zero” we are thinking of a small $\delta$, not a large one. Intuitively speaking, a set is open if you can “wiggle” around any point $a \in A$ and still be inside $A$.

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2 The key word here is “general” since we can find neighborhoods about many objects other than real numbers, such as neighborhoods about points in higher dimensions and even functions,
Another way of thinking about open sets is every point \( a \in A \) in an open set is surrounded by points in \( A \) and is insulated from the outside.

**Margin Note:** Because of the matching of real numbers with points on the number line, we often refer to real numbers as “points.”

**Example 1 (Open Sets)**

a) open intervals \((a, b) = \{ x \in \mathbb{R} : a < x < b \}\) you studied in calculus like \((0, 1), (-1, 3)\) are open sets

b) the following unbounded open intervals are open sets

\[
(a, \infty) = \{ x \in \mathbb{R} : x > a \}
\]

\[
(-\infty, b) = \{ x \in \mathbb{R} : x < b \}
\]

What does it mean for a set not to be open? To answer that question, we negate the definition of an open set, getting

\[
A \text{ is not open } \iff \neg (\forall a \in A)(\exists \delta > 0)(N_\delta(a) \subseteq A)
\]

\[
\iff (\exists a \in A)(\forall \delta > 0)(N_\delta(a) \nsubseteq A)
\]

In other words, there exists at least one point \( a \in A \) “right on the boundary” of \( A \) that is not “insulated” from the outside.

The interval \((a, b]\) is not open since if the point \( x = b \) (which belongs to \((a, b]\)) is wiggled any amount of distance (say \(10^{-100}\)) to the right, it will be outside \((a, b]\). In other words for any \( \delta > 0 \), we have \( N_\delta(b) = (b - \delta, b + \delta) \nsubseteq (a, b]\).

The most important property of open sets relates to their union and intersection.

**Theorem 1 (Main Theorem of Open Sets)** The union of any collection (possibly infinite) of open sets is open. The intersection of a finite number of open sets is open.

**Proof**

(Union of Open Sets) Let \( \{ A_\alpha \}_{\alpha \in \Delta} \) be a family of open sets. To show \( \bigcup_{\alpha \in \Delta} A_\alpha \) is open, we let \( a \in \bigcup_{\alpha \in \Delta} A_\alpha \) which means \( a \in A_\alpha \) for some \( \alpha \in \Delta \). But by
assumption all the sets $A_a$ are open which implies $\exists \delta > 0$ such that $a \in N_{\delta}(a) \subseteq A_a$. But $A_a \subseteq \bigcup_{a \in \Delta} A_a$ and so $a \in \bigcup_{a \in \Delta} A_a$. Hence, the union is open.

*(Intersection of Open Sets)* Let $\{A_k\}_{k=1}^n$ be a finite family of open sets and let $a \in \bigcap_{k=1}^n A_k$ which means $a \in A_k$ for each $k = 1, 2, ..., n$. But each $A_k$ is assumed open so there exists a $\delta_k > 0$ such that $a \in N_{\delta_k}(a) \subseteq A_k$. We now pick $\delta = \min\{\delta_k : k = 1, 2, ..., n\} > 0$ which satisfies

$$a \in \bigcap_{k=1}^n N_{\delta_k}(a) \subseteq \bigcap_{k=1}^n A_k$$

which proves the intersection $\bigcap_{k=1}^n A_k$ is open. □

**Example 2** (Unions and Intersections of Open and Closed Sets)

The following examples illustrate the fact that the union of open sets is open, but an infinite intersection of open sets need not be open.

a) $\bigcup_{n=1}^{\infty} \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right) = (-1, 1)$ (open)

b) $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ (not open)

c) $\bigcap_{n=1}^{5} \left(-\frac{1}{n}, \frac{1}{n}\right) = \left(-\frac{1}{5}, \frac{1}{5}\right)$ (open)

**Closed Sets**

The concept of open sets leads us to what might be called the opposite of an open set, a *closed* set.

**Definition:** A set $A \subseteq \mathbb{R}$ is **closed** if its complement $\bar{A}$ is open.

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3 It is necessary that the number of open sets be *finite*, else the values of $\delta_k$ might *not* have a minimum value.
Example 3 (Closed Sets)

a) The closed intervals $A = [a, b]$ studied in calculus are closed sets since their complements $\overline{A} = (\infty, a) \cup (b, \infty)$ are open. For example the closed intervals $[0, 1], [2, 3]$ are closed sets.

b) The unbounded closed intervals $A = [a, \infty), B = (-\infty, b]$ are closed since their complements $\overline{A} = (\infty, a), \overline{B} = (b, \infty)$ are open. For example $[0, \infty), (-\infty, 0]$ are closed sets.

c) Any singleton set $\{a\}$ is a closed set since its compliment $(-\infty, a) \cup (a, \infty)$ is open. In fact any finite set $\{a_1, a_2, ..., a_n\}$ is closed since its compliment is the union of open intervals (right?), which by Theorem 1 is an open set.

Keep in mind not all sets of are open or closed: the sets $A = (0, 1]$ and $B = [-3, 2)$ are neither open nor closed.

Two sets that are open are the entire real line $\mathbb{R}$ and the empty set $\emptyset$. The set $\mathbb{R}$ satisfies the test of being open (every real number contains a neighborhood in the set), and the empty set is open vacuously because there is no point $a \in \emptyset$ to check. But the complement of $\mathbb{R}$ is $\emptyset$ and hence $\emptyset$ must also be closed. And since $\emptyset$ is open its complement $\mathbb{R}$ must be closed. We conclude $\mathbb{R}$ and $\emptyset$ are both open and closed. In fact these are the only two sets of real numbers that are both open and closed, all other sets are either open, closed, or neither.

We have seen that the intersection of an arbitrary collection of open sets is open, and that the finite intersection of open sets is open. We now ask what are the corresponding properties for closed sets. The following theorem answers this question and shows the “dual” nature of this property.

Theorem 2 (Main Theorem of Closed Sets) The intersection of an arbitrary family or collection (possibly infinite) of closed sets is closed. The union of a finite number of closed sets is closed. The proof is based on DeMorgan’s laws for sets which is left to the reader. (See Problem 4).

Margin Note: Open and closed sets are at the center of point set topology. In general topology, open sets are defined as any collection of subsets of a larger set (say the real line) that is “closed” under unions and finite intersections. By choosing more or less open sets, a variety of different
types of convergence is possible. The more open sets the more difficult for sequences to converge, the fewer open sets the more likely a sequence will converge.

Example 4 (Unions and Intersections of Closed Sets) The following examples illustrate that the intersection of closed sets is closed, but union of closed sets may not be closed, unless it is the union of a finite set.

a) \( \bigcap_{n=1}^{\infty} \left[ \frac{1}{n}, \frac{1}{n} \right] = \{0\} \) (closed)

b) \( \bigcup_{n=1}^{\infty} \left[ -1 + \frac{1}{n}, 1 - \frac{1}{n} \right] = (-1,1) \) (not closed)

c) \([0,2] \cup [1,4] = [0,4]\) (closed)

Interior, Exterior, and Boundary of a Set

Related to open and closed sets is the concept of interior, exterior, and boundary points of a set. 

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4 We are studying very basic topology of the real numbers, which is the study of closeness. In general a topology on a set (such as the real numbers) is any family of subsets, whose members are called open sets, of the set that is closed under unions and finite intersections (exactly the two properties of our open sets). Open sets defined in this section via neighborhoods is only one topology for the real numbers, called the usual topology for the reals. By having more open sets in the topology it “separates” points and makes it harder for sequences to converge. In other words, the topology placed on a set dictates the metric geometry of the space.
Definition:

**Interior Point:** Let \( A \subseteq \mathbb{R} \) be a set of real numbers. A point \( a \in A \) is an interior point of \( A \) if and only if there exists a \( \delta > 0 \) such that \( a \in N_\delta(a) \subseteq A \). We denote the set of interior points of a set \( A \) by \( \text{Int}(A) \).

![Diagram of interior point](image)

**Boundary Point:** A point \( x \) is a boundary point of \( A \) if and only if for any \( \delta > 0 \) the \( \delta \)-neighborhood of \( x \) intersects both \( A \) and the complement of \( A \). That is, \( x \) is a boundary point of \( A \) iff

\[
N_\delta(x) \cap A \neq \emptyset \quad \text{and} \quad N_\delta(x) \cap \overline{A} \neq \emptyset.
\]

A boundary point of a set may or may not belong to the set. We denote the set of boundary points of a set \( A \) by \( \text{Bdy}(A) \).

![Diagram of boundary points](image)

**Margin Note:** Intuitively, a point in a set is an interior point if it is not “right on the edge” of the set. Boundary points are “right on the edge” of the set, they are both close to points inside and outside of the set.
Example 4

- Every point of an open set is an interior point. For example every point of \((0,1)\) is an interior point. The open interval \((0,1)\) has boundary points \(\text{Bdy}((0,1)) = \{0,1\}\).

- The interval \([a,b]\) has interior points \((a,b)\) and boundary points \(\{a,b\}\).

- The interior of the rational numbers \(\mathbb{Q}\) is the empty set, and the boundary is \(\text{Bdy}(\mathbb{Q}) = \mathbb{R}\).
Problems

1. Tell if the following sets are open, closed, both, or neither.
   
   b) \((-1, 0) \cup (0, 1)\)
   c) \([0, 1] \cup [0, 1]\)
   d) \([-1, 0] \cup (0, 1)\)
   e) \(\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\)
   f) \(\mathbb{Q}\) = rational numbers
   g) \(A = \{0, 1, 2, \ldots, 100\}\)
   h) \(\{x : |x-1| > 3\}\)
   i) \(\{x : |x-1| < 2\}\)
   j) \(\{x : x^2 > 0\}\)
   k) \(\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, -\frac{1}{n}\right)\)
   l) \(\bigcap_{k=1}^{\infty} \left[0, \frac{1}{k}\right]\)

2. For the following sets find the interior points and boundary points.
   
   a) \(\emptyset\)
   b) \((0, 1) \cup \{2\}\)
   c) \(\{0, 1, 2, 3, 4, 5\}\)
   d) \(\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}\)
   e) \([0, \infty)\)
   f) \([-1, 0) \cup [0, 1)\)

3. (Finite Sets Closed) Show that the finite set \(A = \{-2, -1, 0, 1, 2\}\) is closed by finding its complement and showing that the complement passes the test of being an open set.

4. (Closed Sets) Show that the union of a any family or collection (possibly infinite) of closed sets is closed. The intersection of a finite number of closed sets is closed. Hint: Use Theorem 1 in the text and DeMorgan’s Laws.
5. **(Cantor Set)** Let \( I = [0,1] \). Remove the open middle third \( \left( \frac{1}{3}, \frac{2}{3} \right) \) and call \( A_1 \) the set that remains; that is
\[
A_1 = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right].
\]

Now remove the open third intervals from each of these two parts of \( A_1 \), and call the remaining part \( A_2 \). Thus
\[
A_2 = \left[ 0, \frac{1}{9} \right] \cup \left( \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right).
\]

Continuing in this manner, remove the open middle third of each segment in \( A_k \) and call the remaining set \( A_{k+1} \). Note that we will get
\[
A_1 \supset A_2 \supset A_3 \supset \cdots A_k \supset \cdots
\]

Continue this process indefinitely, always removing the middle third of existing segments. See Figure 1. The end set of this infinite process is called the Cantor set, and is defined as
\[
C = \bigcap_{k=1}^{\infty} A_k.
\]

Are there be any points left in the Cantor set? Show the Cantor set is not empty and is closed\(^5\).

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\(^5\) The Cantor set has a variety of interesting mathematical properties; has no interior, every point is an accumulation point, is uncountable but at the same time has total “length” (measure) zero.
6. (Counterexample) Find an example of a family of open sets whose intersection is not open.

7. (Counterexample) Find an example of a family of closed sets whose union is not closed.