Section 5.2 *The Complete Ordered Field: The Real Numbers*

**Purpose of Section** We present an axiomatic description of the real numbers as a complete ordered field. The axioms which describe the arithmetic of the real numbers form an algebraic field. The order axioms combined with the field axioms form a mathematical structure known as an ordered field, and finally the completeness axiom are combined with the ordered field axioms to give us the real number system.

**Introduction**

So what are the real numbers? Quite simply they were introduced originally as an idealization of the concept of length (and time) where we envision real numbers as points on a continuous line that extends indefinitely in both directions. Although this is a good intuitive interpretation, the goal in this chapter is to strip away everything you know about the real numbers and start afresh. This is not an easy thing to do since all the knowledge and mental imagery you have created for understanding the subject are firmly entrenched in your mind. But if you are up to wiping the slate clean and start anew, we will introduce you to a new mathematical concept, known affectionately by mathematicians as the “complete, ordered field”, which, for the fun of it we call \( \mathbb{R} \). By building up the axioms of the real numbers, you should have a deeper understanding of them than as “points on a very long line.”

There are three types of axioms required to form what we know as real numbers. First, there are the arithmetic axioms, called the field axioms, which provide the rules for adding, subtracting, multiplying and dividing. Secondly, there are the order axioms, which allow one to compare sizes of real numbers like \( 2<3 \), \( 4>0 \) and \( -3<0 \), and so on. And lastly there is an axiom, called the continuity axiom, which gives the real numbers that special quality that allows us to think of real numbers as “flowing” continuously, with no gaps along the way, on the real line from the infinitely small to the infinitely large.

So let us begin our quest to find the holy grail of real analysis.

**Arithmetic Axioms for Real Numbers**

We begin by defining a set \( \mathbb{R} \), but don’t think of \( \mathbb{R} \) as the real numbers yet, we will when we have equipped \( \mathbb{R} \) with a certain set of axioms. We now define two functions from \( \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), one called the addition function, the other the multiplication function. The addition function assigns to a pair \((a,b)\) of numbers in \( \mathbb{R} \) a new element of \( \mathbb{R} \) called the sum of \( a \) and \( b \) and denoted
by $a + b$. The multiplication function assigns to each pair of elements in $\mathbb{R}$ a new element in $\mathbb{R}$ called the product of $a$ and $b$ and denoted by $a \cdot b$ or more often simply $ab$. These operations are called closed operations since when $a, b \in \mathbb{R}$ so are $a + b$ and $ab$.

These axioms have passed the test of time and are now chiseled in stone in the laws of mathematics and form an an algebraic system called a field¹ (or an algebraic field), which is summarized as follows.

### Field Axioms

A field is a set which, we call $\mathbb{R}$, with two binary operations, called + and $\cdot$, where for all $a$, $b$, and $c$ in $\mathbb{R}$, the following axioms hold⁴.

<table>
<thead>
<tr>
<th>Addition Axioms</th>
<th>Name of Axiom</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $a + (b + c) = (a + b) + c$</td>
<td>(i) associativity of addition</td>
</tr>
<tr>
<td>(ii) $a + b = b + a$</td>
<td>(ii) addition commutes</td>
</tr>
<tr>
<td>(iii) There exists a unique element $0 \in \mathbb{R}$ such that $a + 0 = a$.</td>
<td>(iii) There exists a unique additive identity denoted by &quot;0&quot;.</td>
</tr>
<tr>
<td>(iv) For any $a \in \mathbb{R}$ there exists a unique $x \in \mathbb{R}$ that satisfies $a + x = 0$.</td>
<td>(iv) Every element has a unique additive inverse.</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Multiplication Axioms</th>
<th>Name of Axiom</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $a (bc) = (ab)c$</td>
<td>(i) associativity of multiplication</td>
</tr>
<tr>
<td>(ii) $ab = ba$</td>
<td>(ii) multiplication commutes</td>
</tr>
<tr>
<td>(iii) There exists a unique element $1 \in \mathbb{R}$ (1 ≠ 0) such that $a1 = a$.</td>
<td>(iii) There exists a unique multiplicative identity denoted by &quot;1&quot;.</td>
</tr>
<tr>
<td>(iv) For any $a \in \mathbb{R}$ (except $a = 0$) there exists a unique $y \in \mathbb{R}$ such that $ay = 1$.</td>
<td>(iv) Every nonzero element has a unique multiplicative inverse.</td>
</tr>
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<tr>
<th>Distributive Axiom</th>
<th>Name of Axiom</th>
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<tr>
<td>$a(b + c) = ab + ac$</td>
<td>Multiplication distributes over addition.</td>
</tr>
</tbody>
</table>

¹ Modern algebra or abstract algebra, which is distinct from elementary algebra as taught in schools, is a branch of mathematics that studies algebraic structures, such as groups, rings, fields, modules, vector spaces and other algebraic structures.
Conventions and Notation:

In addition to the above axioms, we make the following conventions:

1. The associative axioms for both addition and multiplication say it doesn’t matter where parenthesis are placed. In other words, we can write $a + b + c$ for $a + (b + c)$ or $(a + b) + c$. The same holds for multiplication, we can write $abc = a(bc) = (ab)c$.

2. The unique additive inverse of an element $a$ (relative to the additive identity 0) is normally denoted by $-a$, hence we have $a + (-a) = 0$. The multiplicative inverse of $a$ (relative to the multiplicative identity 1) is denoted by $a^{-1}$ (provided $a \neq 0$) and often written $1/a$. Hence $aa^{-1} = a(1/a) = 1$.

Two other operations of subtraction and division can be defined directly from addition and multiplication by

\[
\begin{array}{ll}
\text{subtraction:} & a - b = a + (-b) \quad \text{(read a minus b)} \\
\text{division:} & \frac{a}{b} = ab^{-1} \quad \text{(for } b \neq 0) \quad \text{(read a divided by b)}
\end{array}
\]

Using the basic axioms for a field, we can now see how the computational rules for arithmetic can be carried out. We define $2 = 1 + 1, 3 = 2 + 1, 4 = 3 + 1,$ and so on. If we now define the natural numbers as $\{1, 2, 3, \ldots\}$, then since $1 > 0$, it follows that $0 < 1 < 2 < 3 < \ldots$. In other words, the natural numbers are ordered in the way we learned since grade school. We can also prove something you learned in the second grade:

\[
\begin{align*}
4 &= 3 + 1 \quad \text{(definition)} \\
&= (2+1) + 1 \quad \text{(definition)} \\
&= 2 + (1+1) \quad \text{(associativity)} \\
&= 2 + 2 \quad \text{(definition)}
\end{align*}
\]

Margin Note: A field is an algebraic system where you can add, subtract, multiply and divide (except by 0) in the same manner you did as a child. As a child you were taught these were “properties” of numbers. But they are not their properties, they are the rules of engagement of the real numbers. A subtle, but important point.

\[\text{We call the field } \mathbb{R} \text{ since we are concentrating on the real numbers, but keep in mind there are many examples of an algebraic field.}\]
We know what you are thinking; you have known all this since 3rd grade. If your argument is that the axioms are simple and elementary, that is no argument at all. Axioms are supposed to be self-evident. That’s the test of a good axiom system. The question you ask is: what kind of theorems can be proven from the axioms, and the answer is there are many and many are not trivial. Just ask yourself, are these the simplest axioms you can imagine for a system of arithmetic, where you can add, subtract, multiply and divide? Do you need any more axioms to perform the operations you want? Can you get by for fewer axioms in the sense that some of the axioms can be proven from the others and hence ambiguous? These are not trivial questions and their answers are even less so. There are other axiom systems that allow you to perform operations on elements of a set, such as groups, rings, integer domains, and so on, but the axiom system you studied as a child is an algebraic field.

**Margin Note:** The American geometer Oswald Veblen (1880-1960) once said the test of a good axiom system lies in the theorems the axioms produce.

**Examples of Fields other than \( \mathbb{R} \)**

1. **Boolean Field:** Let \( F_2 = \{0, 1\} \) and define addition (+) and multiplication (×) by the following table.

   \[
   \begin{array}{c|c|c}
   + & 0 & 1 \\
   \hline
   0 & 0 & 1 \\
   1 & 1 & 0 \\
   \end{array}
   \begin{array}{c|c|c}
   \times & 0 & 1 \\
   \hline
   0 & 0 & 0 \\
   1 & 0 & 1 \\
   \end{array}
   \]

   The set \( A \) with these arithmetic operations is a field. We leave it to the reader to check all the properties a field must possess.

2. **Complex Numbers** \( \mathbb{C} \): The complex numbers \( a + bi \), where \( a, b \) are real numbers and \( i = \sqrt{-1} \), where addition and multiplication are defined in the usual manner.

3. **Rational numbers** \( \mathbb{Q} \): The rational numbers where addition and multiplication are defined in the usual way.

There are many other examples of fields studied by mathematicians, including the Galois finite fields, \( p \)-adic number fields, and fields of functions, such as meromorphic and entire functions.

We now come to the second group of the three types of axioms required to describe the number system we desire, the order axioms.
Order Axioms: Ordering the Reals

Our goal now is to order the real numbers\(^3\) in a way consistent with the arithmetic properties of a field. For example, for two rectangles with the same base \(c\), we require the taller one to have the larger area, which basically says if \(b > a\) and \(c > 0\) then \(bc > ac\). See Figure 1.

![Figure 1](image)

We also are desirous to build into the system the concept that the real numbers are an abstract version of a ruler which extends indefinitely in both directions; hence we divide the elements of \(\mathbb{R}\) into three disjoint sets

\[
\mathbb{R} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+
\]

where \(\mathbb{R}_-\) is called the negative numbers, \(\mathbb{R}_+\) is called the positive numbers and 0 is the additive identity of the field. We then require the additive inverse \(-x\) of \(x\) to satisfy

\[
x \in \mathbb{R}_+ \iff -x \in \mathbb{R}_- \quad \text{and} \quad x \in \mathbb{R}_- \iff -x \in \mathbb{R}_+.
\]

Other than that, we require that for any two positive numbers \(a,b\) (i.e. \(a,b \in \mathbb{R}_+\)) their sum \(a+b\) and product \(ab\) are also positive. That’s it!

For convenience, when \(x \in \mathbb{R}_+\) we write \(x > 0\) and say \(x\) is greater than 0. Likewise, when \(x \in \mathbb{R}_-\) we write \(x < 0\) and say \(x\) is less than 0. Also, when

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\(^3\) As we have been saying over and over, we are using the real numbers as a model for our abstract discussion. There are many ordered fields that are not the real numbers. We keep calling our ordered field the real numbers inasmuch as that is the focus of our discussion.
We write \( a > b \) or \( a = b \), we denote this as \( a \geq b \) or \( b \leq a \).

So let’s put everything we have thus far in a formal mathematical system. We call it an ordered field.

**Definition:** An algebraic field \( \{\mathbb{R}, +, x\} \) is an ordered field if \( \mathbb{R} \) is an algebraic field that satisfies the following conditions:

I. For each \( x \in \mathbb{R} \), exactly one of \( x \leq 0 \), \( x = 0 \), or \( x \geq 0 \) is true.
II. For \( x > 0 \) and \( y > 0 \), then \( x + y > 0 \).
III. If \( x > 0 \) and \( y > 0 \), then \( xy > 0 \).

**Theorem 1** The relation "\( \leq \)" (read less than or equal to) is a total order on \( \mathbb{R} \). That is for any elements \( a, b, c \) in \( \mathbb{R} \), either \( a \leq b \) or \( b \leq a \), as well as the following RAT axioms hold.

(i) Reflexive: \( a \leq a \).
(ii) Anti-Symmetric: If \( a \geq b \) and \( a \leq b \), then \( a = b \).
(iii) Transitive: If \( a \leq b \) and \( b \leq c \), then \( a \leq c \).

**Proof:**

Clearly \( a \leq b \) or \( b \leq a \) since if \( a \leq b \) is not true, then \( b - a \) is not greater than or equal to zero, hence by axiom I is negative. But then \( -(b - a) = a - b \) is positive or \( a \geq b \). Thus either \( a \leq b \) or \( b \leq a \).

i) Reflexive: \( (a \leq a) \)

\( a \leq a \) means \( a - a = 0 \) which, from the field axioms, is true.

ii) Anti-Symmetric: \( [(a \leq b) \wedge (a \geq b)] \Rightarrow a = b \)

From Axiom 1, we know that any real number is either positive, negative or zero. But \( a \leq b \) says \( b - a \) is positive or zero, and \( a \geq b \) says \( a - b \) is positive or zero, or equivalently \( b - a = -(a - b) \) is negative or zero. Hence \( b - a = 0 \), or \( a = b \).

\[ \]
iii) Transitive: \([ (a \leq b) \land (b \leq c) ] \Rightarrow a \leq c\)

First observe that if \(a = b \) or \(a = c\), the result is trivial, since if we replace \(a\) by \(b\), the desired conclusion becomes \([ (b \leq b) \land (b \leq c) ] \Rightarrow b \leq c\) which states the tautology \(b \leq c \Rightarrow b \leq c\). Now, if we let \(a = c\), the desired conclusion becomes \([ (c \leq b) \land (b \leq c) ] \Rightarrow c \leq c\), which has a true conclusion hence the implication is true.

We now come to what we called the “holy grail” of the real numbers. Although all three collections of axioms are necessary to describe the real numbers, it might be said that the “completeness” axiom is what we think of when we think of the real numbers.

**The Completeness Axiom**

If we were to stop at the arithmetic and ordered axioms for the real numbers, we would be throwing out that special ingredient that makes the real numbers that *continuum* of numbers we envision when we think of the real number system. There are many examples of ordered fields that are *not* the real numbers, and everyone of these algebraic systems fields suffer by having “gaps” between their elements. For example, the rational numbers, i.e. numbers of the form \(Q = \{ p/q : p, q \in \mathbb{N}, q \neq 0 \}\) is an ordered field, but it has gaps, two gaps being the solutions of \(x^2 = 2\), which are \(x = \pm \sqrt{2}\), which we know are not rational numbers. What we need is an axiom that “fills in” these gaps and that is where the *completeness* (or *continuum*) axiom comes into play.

An interesting aspect of the completeness (or continuum) axiom is that over the years mathematicians have found *several* axioms, all called the completeness axiom, that are logically equivalent. Thus, it might be said we are blessed by being able to select any one of them. In this book what is called the *least upper bound* axiom as our “completeness representative” because several interesting concepts can be gleaned by working with it, and, it is easy to understand. Before stating the axiom, however, we must introduce a few important ideas.

**Sup and Max (Inf and Min)**

We use the four intervals in Figure 2 as a prop for familiarizing ourselves with the concepts of what are called *lub, glb, sup, and inf.*
The intervals \((a, b), [a, b], [a, b],\) and \((a, b]\) are all bounded, both above and below. Bounded above simply means there is at least one number greater than or equal to all the elements in the set. The number \(b\) (or any number larger) is an upper bound for each of the four intervals in Figure 2. Likewise, a lower bound for a set is a number less than or equal to all the elements in the set; the number \(a\) is a lower bound for each of the above sets. Of course, not all sets are bounded: the set \([1, \infty)\) is bounded below but not above, and \((\neg \infty, \infty)\) is not bounded above nor below. Also note that \(b\) is the maximum of all the numbers in the intervals \([a, b]\) and \((a, b]\), whereas the intervals \((a, b)\) and \([a, b)\) do not contain a maximum since no matter what number is chosen as the maximum there is always a larger number halfway between your choice and \(b\). The same arguments hold for minimum values, the two intervals \([a, b], [a, b)\) have a minimum value whereas the intervals \((a, b), (a, b]\) do not have minimum values.

So what is the meaning \(\text{lub}(A)\) and \(\text{glb}(A)\) in Figure 2? Two of the sets contain their maximum and two do not. However (and this is the important part), for each of the four intervals, the set of upper bounds, which is \([b, \infty)\)
for each of the four intervals, always contains its minimum value, which is \( b \) in every case. In the intervals \((a, b), [a, b]\) where \( b \) does not belong to the interval, we call this value least upper bound or supremum of the set, and denote this value by \( \text{sup}(A) \) or \( \text{lub}(A) \). For the two sets \([a, b]\) and \((a, b]\) that have a maximum value, the least upper bound is the same as the maximum. For the sets \((a, b)\) and \([a, b]\) that do not have maximum values, the least upper bound \( b \) is a kind of “surrogate” for the maximum.

The same principle holds for lower bounds. The set of lower bounds for any set \( A \) bounded below always has a largest value and this value is called the greatest lower bound or infimum of \( A \) and denoted by \( \text{glb}(A) \) or \( \text{inf}(A) \).

**Definition** Let \( A \) be a set bounded above in an ordered field. The number \( L \) is the least upper bound or supremum of \( A \) if

- \( L \) is an upper bound of \( A \), i.e. \( L \geq x \) for all \( x \in A \).
- if \( u \) is any upper bound for \( A \), then \( L \leq u \).

Likewise the number \( G \) is the greatest lower bound or infimum of \( A \) if

- \( G \) is a lower bound of \( A \), i.e. \( G \leq x \) all \( x \in A \).
- if \( l \) is any lower bound for \( A \), then \( G \geq l \).

The least upper bound of a set is denoted \( \text{lub}(A) \) or \( \text{sup}(A) \), and the greatest lower bound is denoted \( \text{glb}(A) \) or \( \text{inf}(A) \).
This leads us to the completeness axiom for our set \( \mathbb{R} \), which we have endowed with field and order axioms and thus is an ordered field (we are almost ready to call this set the real numbers). The last set of axioms we assign to \( \mathbb{R} \) (actually only one axiom) is called the completeness axiom, which we have used the least upper bound version.

**Completeness Axiom: Least Upper Bound Axiom**

If any non-empty set of \( \mathbb{R} \) that is bounded above has a smallest upper bound, then it satisfies the completeness axiom.

We are now (finally) ready to define the real numbers.

**Definition:** The real number system \( \mathbb{R} \) is an ordered field that satisfies the completeness axiom, that is, a complete ordered field. Stated another way it is a set \( \mathbb{R} \) that satisfies the axioms of an algebraic field, the order axioms, and the completeness axiom.

The least upper bound is necessary since there are ordered fields that do not “look like” the real numbers, the rational numbers being one such example. By also including the completeness axiom, the ordered field behaves exactly like the real line you learned about in the third grade. In other words it has exactly the properties we desire when we try to model or describe points on an infinite line.

When we refer to the real numbers as a complete ordered field, we always say the complete ordered field since there is only one complete ordered field in the sense that any two complete ordered fields are isomorphic. We say that two abstract structures are isomorphic if they have exactly the same mathematical structure and differ in only the symbols used to represent various objects and operations in the system. In the case of two fields, the objects would be \((A,0_A,1_A,+_A,\times_A,\leq_A)\) for one field and \((B,0_B,1_B,+_B,\times_B,\leq_B)\) for another. An isomorphism between these two fields would be a one-to-one correspondence \(f:R\rightarrow S\) satisfying for all \(x,y\in A\):

\[
\begin{align*}
f(0_A) &= f(0_B) \\
f(1_A) &= f(1_B) \\
f(x +_A y) &= f(x) +_B f(y) \\
f(x \times_A y) &= f(x) \times_B f(y) \\
x \leq_A y &\Rightarrow f(x) \leq_B f(y)
\end{align*}
\]

This means that all complete ordered fields are essentially the same, the only difference being a labeling of objects.
Note: The set of rational numbers $A = \{q \in Q : q^2 < 2\}$ does not satisfy the completeness axiom; it is bounded above but does not have a least upper bound.
Problems

Algebraic Fields: Problems 1–3 assume \(a, b\) are members of an algebraic field.

1. Use only the axioms of an algebraic field to prove \((-a) = a\). \textbf{Hint:} \(a\) and \(-(-a)\) are both solutions of \(x + (-a) = 0\).

2. Use only the axioms of an algebraic field to prove \(a \cdot 0 = 0\). \textbf{Hint:} \(a \cdot 0 + a \cdot 0 = a(0 + 0) = a \cdot 0 = a \cdot 0 + 0\) and so \(a \cdot 0\) and 0 are both solutions of \(x + a \cdot 0 = a \cdot 0\).

3. Use only the axioms of an algebraic field to prove \((ab)^{-1} = b^{-1}a^{-1}\), where \(a, b\) are not 0. \textbf{Hint:} Show both \((ab)^{-1}\) and \(b^{-1}a^{-1}\) are solutions of \(x(ab) = 1\).

Order Axioms: For Problems 4–6 assume \(a, b\) are members of an ordered field.

4. Use only the axioms of an ordered field to prove if \(a > b\) and \(b > c\), then \(a > c\). \textbf{Hint:} Use the facts that \(a - b \in \mathbb{R}_+\), \(c - d \in \mathbb{R}_+\) imply \(a - c \in \mathbb{R}_+\).

5. Use only the axioms of an ordered field to prove if \(a > b\) and \(b \geq c\), then \(a \geq c\). \textbf{Hint:} Use the facts that \(a - b \in \mathbb{R}_+\), \(c - d \in \mathbb{R}_+ \cup \{0\}\) imply \(a - c \in \mathbb{R}_+ \cup \{0\}\).

6. Use only the axioms of an ordered field to prove if \(a > 0\), then \(1/a > 0\). \textbf{Hint:} Use the fact that \(a \cdot (1/a) = 1\), which would contradict the rules of sign if \(1/a \leq 0\).

7. For the following sets \(A\) find (if they exist), \(\max(A), \min(A), \sup(A), \inf(A)\).

   a) \(A = \{1, 3, 9, 4, 0\}\)
   b) \(A = [0, \infty)\)
   c) \(A = \{x \in \mathbb{Q} : 0 \leq x \leq 1\}\)
   d) \(A = [-1, 3]\)
   e) \(A = \{x : x^2 - 1 = 0\}\)
   f) \(A = \{n \in \mathbb{N} : n \text{ divides 100}\}\)
g) \( A = \{ x \in \mathbb{R} : x^2 < 2 \} \)

h) \( A = (-\infty, \infty) \)

i) \( A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\} \)

8. (Algebraic Field) Show that the rational numbers with the operations of addition and multiplication form an algebraic field.

9. (Tiny Field) Show that the set containing only two elements \( \{0, 1\} \) form an algebraic field. In other words the set contains only the additive and multiplicative identities.

10. (Ordered Field) Show that the rational numbers with the operations of addition and multiplication and the usual “less than” order relation "<" forms an ordered field.