Section 5.1 Construction of the Real Numbers: $\mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R}$

**Purpose of Section** Starting with the non negative integers, we construct, in order, the integers, rational numbers and the real numbers, using equivalence relations.

**Introduction**

Every school child knows about real numbers. Phrases such as “all numbers between 0 and 1” or “any length between 2 and 3” are familiar to all of us. We think of real numbers geometrically as points on a line. The idea of a *continuous* range of values has always been the accepted model of the three basic physical measurements of length, mass, and time. However, quantum physicists tell us that in the tiny world of quantum physics, we cannot admit the possibility of continuous observations; namely that all observations must take place at discrete, isolated instances.

However, in mathematics, we *can* observe continuously, at least in our heads. For instance, take a continuum, say from 0 to 1, and calling a variable $x$, we can imagine continuous quantities like $x^2$ or $\sin x$. But a physicist may argue, if a mathematician were to peer closer and closer into the continuum, strange things may be observed, just as it has in physics. It is the purpose of this and the next section to ask ourselves, just what *are* real numbers, and find out what happens when we look at them ... up close.

The path from the natural numbers 1,2,3,... to the real numbers was a journey that took several thousands years. The natural (or counting) numbers probably arose from counting: counting goats, sheep, or whatever possessions one had. Fractions, or rational numbers, are simply a refinement of counting finer units, like $\frac{1}{2}$ bushel of wheat, $\frac{1}{3}$ of a mile, and were well-known to Greek geometers. Although Greek mathematicians routinely used rational numbers, they did not accept zero², or negative numbers as legitimate numbers. To them, numbers represented quantities like length and area, things that are never 0 or negative. Believe it or not, Columbus discovered America before mathematicians discovered negative numbers, or maybe we should say *accepted* them as numbers. Even such prominent Renaissance mathematicians as Cardano in Italy, and Viete in France called negative numbers “absurd” and “fictitious.” But gradually, mathematicians realized the need to enlarge their thinking and address paradoxes like $\frac{1}{2} - \frac{1}{3}$, or the solution

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¹ The word rational is the adjective form of ratio.
² The first occurrence of a symbol used to represent 0 goes back to Hindu writings in India in the 9th century.
of equations like \( x + 3 = 1 \) so negative numbers finally were brought into the family of numbers. But the last step in evolution of real numbers from natural numbers took considerably more thought. One of the great mathematical achievements of the 19th century, was the understanding of what we call the “real numbers”: those numbers which can be expressed in decimal notation, whether the decimal digits stop, go on forever in a repeating pattern, or go on in a pattern that never repeats. So now we arrive at this foundation of real analysis, the real numbers. So what are they?

There are two basic ways to define the real numbers. First, we can play God and bring the “laws” down from the mountaintop, so to speak, where we lay down the rules of the game and say, *Here, these are the real numbers.* This approach would be called the *synthetic* approach, whereby we list a series of axioms, which we feel are the embodiment of what we think a “continuum” should be. On the other hand, we can “construct” the real numbers, much like a carpenter does in building a house. In this approach, we begin with the simplest numbers, like the natural numbers \( 1, 2, 3, \ldots \), and then doing some “mathematical construction” one builds the real numbers. It is this “construction” approach we carry out in this section. The synthetic axiomatic approach will be carried out in the next section.

The Building of the Real Numbers

The construction of the real numbers begins with the simplest of mathematical objects, the non negative integers \( 0, 1, 2, 3, \ldots \). \(^3\)

\[
\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}
\]

then by a series of steps, we construct the integers

\[
\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}
\]

followed by the rational numbers

\[
\mathbb{Q} = \{p / q : p, q \in \mathbb{Z}, q \neq 0\}
\]

and finally the real numbers \( \mathbb{R} \).

**Step 1:** \( \mathbb{N} \cup \{0\} \to \mathbb{Z} \)

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\(^3\) We could just as well start with the natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \) but it is more convenient to include 0 with the natural numbers.
So how does one “construct” the integers $0, \pm 1, \pm 2, \ldots$ from the nonnegative numbers $0, 1, 2, 3, \ldots$? The idea is define integers as pairs of nonnegative integers, where we “think” of each pair $(m, n)$ as representing the difference $m - n$, and thus a pair like $(2, 5)$ would represent $-3$. If we then define addition, subtraction, and multiplication of the pairs $(m, n)$ in a way that is consistent with the arithmetic of the nonnegative integers, we have successfully defined the integers in terms of the nonnegative integers. To carry out this program, we start with the set

$$S = \{(m, n) : m, n = 0, 1, 2, \ldots\}$$

of pairs of nonnegative integers, called grid points, which are illustrated as dots in Figure 1.

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4 The idea behind the equivalence relation is that $(m, n) \equiv (m', n')$ if and only if $m - n = m' - n'$ except we are not allowed to use negative numbers as of now. Hence, we get by this and say the equivalent statement $(m, n) \equiv (m', n')$ if and only if $m + n' = m' + n$.
$(m,n) \equiv (m',n')$ if and only if $m + n' = m' + n$

It is an easy matter to show this relation is an equivalence relation on $S$ (See Problem 1.) and thus it partitions $S$ into (disjoint) equivalence classes, each equivalence class being the grid points on a straight line of the form $n = m + k$, $k = 0, \pm 1, \pm 2, \ldots$. See Figure 1. A few equivalences classes are listed in Table 1, along with their designated representative $(\overline{m})$. Each one of these equivalence classes will represent an integer, equivalence classes of the form $(\overline{m,0})$ will represent positive integers $m$, $(\overline{0,0})$ will be zero, and pairs of the form $(\overline{0,n})$ will represent a negative integers $-n$.

<table>
<thead>
<tr>
<th>Equivalence Class</th>
<th>Integer Correspondence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,2) = {(0,2)(1,3),(2,4),(3,5)\ldots}$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$(0,1) = {(0,1)(1,2),(2,3),(3,4)\ldots}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$(0,0) = {(0,0)(1,1),(2,2),(3,3)\ldots}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$(1,0) = {(1,0)(2,1),(3,2),(4,3)\ldots}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(2,0) = {(2,0)(3,1),(4,2),(5,3)\ldots}$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

Five Equivalence Classes in $S$

Table 1

For example

$(0,2) = -2 \quad (0,1) = -1 \quad (0,0) = 0 \quad (1,0) = 1 \quad (2,0) = 2$

We now define the integers $\mathbb{Z}$ as the collection of equivalences classes of $S$:

$$\mathbb{Z} = \left\{ \ldots (0,3),(0,2),(0,1),(0,0),(\overline{1,0}),(\overline{2,0}),(3,0),\ldots \right\}$$

$$= \left\{ \ldots -3,-2,-1,0,1,2,3,\ldots \right\}$$

or in general

$$\overline{(k,0)} = k \quad \text{(positive integers)}$$

$$\overline{(0,0)} = 0 \quad \text{(zero)}$$

$$\overline{(0,k)} = -k \quad \text{(negative integers)}$$
We now define addition, subtraction, and multiplication of these newly found integers in a manner consistent with the arithmetic of the natural numbers. We define

<table>
<thead>
<tr>
<th>Addition</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p,0)+(q,0)=(p+q,0))</td>
<td>(3+2=5)</td>
</tr>
<tr>
<td>((0,p)+(0,q)=(0,p+q))</td>
<td>(-5+(-2)=-7)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subtraction</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p,0)-(q,0) = \begin{cases} (p-q,0) &amp; p &gt; q \ (0,q-p) &amp; q &gt; p \end{cases})</td>
<td>(5-2=3)</td>
</tr>
<tr>
<td>(3-5=-2)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p,0)(q,0)=(pq,0))</td>
<td>(3\times2=6)</td>
</tr>
<tr>
<td>((0,p)(0,q)=(pq,0))</td>
<td>(-2\times-3=6)</td>
</tr>
<tr>
<td>((0,p)(q,0)=(0,pq))</td>
<td>(-2\times3=-6)</td>
</tr>
</tbody>
</table>

**Step 2** \(Z \rightarrow \mathbb{Q}\) The next step is to construct the rational numbers from equivalence classes of integers, and we do this by “thinking” of each pair of integers \((m,n)\) as the ratio \(m/n\), which translates into defining the equivalent relation on the Cartesian product \(Z \times (Z - \{0\})\):

\[
(m,n) \equiv (m',n') \iff mn' = m'n
\]

We leave it to the reader to show \(\equiv\) is an equivalence relation (See Problem 2.) The equivalence classes for this equivalence relation are the grid points on straight lines passing through the origin as illustrated in Figure 2.
Equivalence Classes Defining Rational Numbers
Figure 2

A few equivalence classes with designated representative are

<table>
<thead>
<tr>
<th>Equivalence Class</th>
<th>Rational Correspondence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\overline{(1,2)}) = {(1,2)(2,4),(3,6)\ldots}</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(\overline{(1,1)}) = {(1,1),(2,2),(3,3)\ldots}</td>
<td>1</td>
</tr>
<tr>
<td>(\overline{(3,5)}) = {(3,5),(-3,-5),(6,10)\ldots}</td>
<td>(\frac{3}{5})</td>
</tr>
<tr>
<td>(\overline{(5,2)}) = {(5,2)(10,4),(15,6)\ldots}</td>
<td>(\frac{5}{2})</td>
</tr>
</tbody>
</table>

Five Equivalence Classes in \(\mathbb{Z} \times (\mathbb{Z} - \{0\})\)

Table 1
We now define a rational number $\frac{p}{q}$ as

\[
\frac{p}{q} = (p, q), \quad p, q \in \mathbb{Z}, \; q \neq 0.
\]

and their arithmetic operations as

<table>
<thead>
<tr>
<th>Operation</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>$\frac{3}{2} + \frac{5}{7} = \frac{6}{35}$</td>
</tr>
<tr>
<td>Subtraction</td>
<td>$\frac{2}{3} - \frac{1}{5} = \frac{7}{15}$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$\frac{3}{5} \cdot \frac{2}{3} = \frac{6}{35}$</td>
</tr>
</tbody>
</table>

Construction of the Rational Numbers from Pairs of Integers

Table 2

**Step 3: Dedekind Cuts $\mathbb{Q} \rightarrow \mathbb{R}$**

There are various ways to define the real numbers and each has its advantages and disadvantages. It is well known (See Problem 1) that for decimal representations of rational numbers, there will always be repeating blocks of digits. For example $\frac{1}{3} = 0.333333\ldots$ repeats in blocks of 1, starting at the first digit, whereas the rational number $\frac{7}{22} = 0.3181818\ldots$ (often written $0.\overline{318}$) has repeating blocks of 2, starting at the second digit. In fact, repeating decimal digits defines the rational numbers. If a decimal expansion does not repeat, the number is not rational, which we call irrational, like $\sqrt{2} = 1.414213\ldots$. The net result is that the real numbers can be defined as all positive and negative decimal expansions, with repeating and nonrepeating digits, one group called rational numbers, the other irrational.

The disadvantage of the above definition of real numbers is that decimal expansions doesn’t relate to points on a line. To relate real numbers to a

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5 In order that decimal expansions be unique, one agrees to represent all nonterminating blocks of 9s, like 0.2399999\ldots by 0.2400000\ldots.

6 The word rational is the adjective form of the word “ratio.”
continuum, there are two basic approaches, one due to Cantor and the other by his good friend and supporter, Richard Dedekind. Roughly, Cantor’s approach defines the real numbers as “limits” of sequences of rational numbers, like

\[ 3, \, 3.1, \, 3.14, \, 3.141, \, 3.1415, \, 3.14159, \, 3.141592, \ldots \rightarrow \pi \]

But this approach, while having an intuitive appeal, leads us into the study of sequences, convergence, null sequences, and other ideas from real analysis we have not introduced, hence we follow the approach of Dedekind, who made the fundamental observation there were “gaps” in the rational numbers and he wanted to fill them.

**Historical Note:**
The German mathematician Richard Dedekind (1831=1916) was one of the greatest mathematicians of the 19\textsuperscript{th} century, as well as one of the greatest contributors to number theory and abstract algebra. His invention of ideals in ring theory and his contributions to algebraic numbers, fields, modules, lattice, etc were crucial in the development of modern algebra. He was also a major contributor to the foundation of mathematics, his definition of the real numbers by means of Dedekind cuts, and his formulation of the Dedekind-Peano axioms were important for the early development of set theory.

The rational numbers have many desirable properties. There are an infinite number of them (\( \mathbb{Q} \) to be exact), one can add, subtract, multiply, divide them, unlike the integers where you can not always divide\(^7\). Also, they are dense, meaning between any two rational numbers there is another rational number (just take the average of the two numbers), in fact there are an infinite number of rational numbers between any two rational numbers. What is not desirable, however, is that there are gaps in them, say at \( \sqrt{2} \) and at other points (in fact an uncountable infinite number of points). The idea is to “fill in” those gaps, getting a new system of numbers (real numbers) which can be placed in a one-to-one correspondence with points on a continuous line.

Dedekind’s idea is surprisingly simple, so simple that beginning students often are left with a sense of “what was all that about?” after seeing it\(^8\). Dedekind’s basic motivation is based on the simple fact that a real number \( \alpha \) is completely determined by the rational numbers less than \( \alpha \) and the rational numbers greater than \( \alpha \). Following this general principle, Dedekind observed that on a line consisting of rational numbers\(^9\), any rational number \( r \), say \( r = 0.7 \), divides the line into two disjoint parts, the lower set \( L \) and the upper set \( U \), defined by

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\(^7\) \( \sqrt{2} \) does not exist for integers.

\(^8\) In order that you can see what is coming, the real numbers will be defined as all Dedekind cuts.

\(^9\) Keep in mind that definition of the real numbers must be completely in terms of rational numbers. We are constructing these new numbers from existing rational numbers.
\[ L = \text{all rational numbers } < r \]
\[ U = \text{all rational numbers } \geq r \]

The pair \((L, U)\) is called a Dedekind cut, where it is important to note that the lower set \(L\) does not have a maximum (rational) number\(^{10}\), but the upper set \(U\) does have a minimum (rational) number, namely 0.7. It is also important to observe that every rational number \(r\) determines a unique Dedekind cut \((L, U)\), and conversely, any Dedekind cut \((L, U)\) with upper set \(U\) having a smallest rational number \(r\), determines this rational number. In other words, we have the obvious observation that there is a one-to-one correspondence between Dedekind cuts at rational numbers and the rational numbers.

No doubt the reader is starting to wonder where all this is going. Just wait. We now define a general Dedekind cut. (Note in the general definition, we haven't said anything about the value of \(\alpha\), the point where the cut occurs.

\(^{10}\) Never use the word “real number” at this point. They have yet to be defined.
**Definition:** A Dedekind cut \((L,U)\) is a subdivision of the rational numbers into two nonempty, disjoint sets \(L\) and \(U\) in such a way that \(L\) has no largest element, and that if \(x \in L, y \in U\), then \(x < y\). The set \(L\) is called the lower interval of the cut, and \(U\) is called the upper interval of the cut.

![Dedekind cut diagram](image)

It is important to note that in this general definition \(\alpha\) is *not* specified as being a rational number.

It was at this point that Dedekind realized there are Dedekind \((L,U)\) where the upper set \(U\) does *not* have a minimum rational number. An example is the cut \((L,U)\), where

\[
L = \text{all negative rational numbers and positive rational numbers whose squares is less than 2} \\
U = \text{positive rational numbers whose squares are greater than or equal to 2}
\]

In this case \(U\) has no smallest rational number since one can find smaller and smaller rational numbers \(r\) for which \(2 \leq r^2\) (pick \(r^2 = 2.1, 2.01, 2.001, \ldots\)), but one cannot pick \(r^2 = 2\) since \(r = \sqrt{2}\) is not rational. So the question arises: where exactly did we make the cut in Figure 4? What is the mystery number?
Dedekind cut where \( U \) has no Smallest Rational Number

Figure 4

It’s simple, Dedekind realized he had filled in a gap in the rational numbers, where in this instance the thing filling in the gap was \( \sqrt{2} \). So this idea motivated Dedekind’s definition of the real numbers.

**Definition:** The real numbers, denoted by \( \mathbb{R} \), is the collection of all Dedekind cuts \((L_U, U)\) on the rational numbers, with each real number being associated with a specific Dedekind cut. If the upper set \( U \) has a smallest rational number, the number associated with the cut is a rational number. If the upper set \( U \) does not have a minimum rational number, the number associated with the cut is an irrational number.

**Properties of the Real Numbers**

Now that the real numbers have been defined, we must define the many properties we desire for the real numbers. Associating real numbers \( a, b \) to their Dedekind cuts

\[
a = (L_a, U_a), b = (L_b, U_b)
\]

where \( L_a, L_b, U_a, U_b \) are the corresponding lower and upper intervals we define the following arithmetic operations and order relation.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a + b = { x + y : x \in L_a, y \in L_b } )</td>
<td>( 2 + 3 = { x + y : x \in L_2, y \in L_3 } = { \text{rational numbers} &lt; 5 } = 5 )</td>
</tr>
<tr>
<td>( a - b = { x - y : x \in L_a, y \in L_b } )</td>
<td>( 2 - 3 = { x - y : x \in L_2, y \in L_3 } = { \text{rational numbers} &lt; -1 } = -1 )</td>
</tr>
<tr>
<td>( a, b &gt; 0 \Rightarrow ab = { xy : x \in L_a, y \in L_b } )</td>
<td>( 2 \cdot 3 = { xy : x \in L_2, y \in L_3 } = 6 )</td>
</tr>
<tr>
<td>( a \leq b \Leftrightarrow L_a \subseteq L_b )</td>
<td>( 2 \leq 3 \Leftrightarrow L_2 \subseteq L_3 )</td>
</tr>
</tbody>
</table>

**The Algebrization of Geometry:** Dedekind cuts (i.e. real numbers), are based on the fundamental property of the Euclidean line that “if all points on a line fall into one of two classes, such that every point in the first class lies to the left of every point in the second class, then there is one and only one point that produces this division. It was this obvious geometric property of a line that inspired Dedekind’s arithmetic formulation of continuity. It might be said that Dedekind separated arithmetic from geometry in the process by creating a purely arithmetic description of the Euclidean line.

\[11\] The values of \( x, y \) in the following definitions are taken as rational numbers.
Problems

1. Find the fraction for each of the following numbers in decimal form.
   a) 0.9999\ldots \quad (0.\overline{9})
   b) 0.23232323\ldots \quad (0.\overline{23})
   c) 0.13513513513\ldots \quad (0.1\overline{3513})
   d) 0.001111\ldots \quad (0.00\overline{1})

2. Show that

   \[(m, n) \equiv (m', n') \text{ if and only if } m + n' = m' + n\]

   defines an equivalence relation on

   \[S = \{(m, n) : m, n = 0, 1, 2, \ldots\}\]

3. Show that

   \[(m, n) \equiv (m', n') \text{ if and only if } mn' = m'n\]

   defines an equivalence relation on \(\mathbb{Z}\).