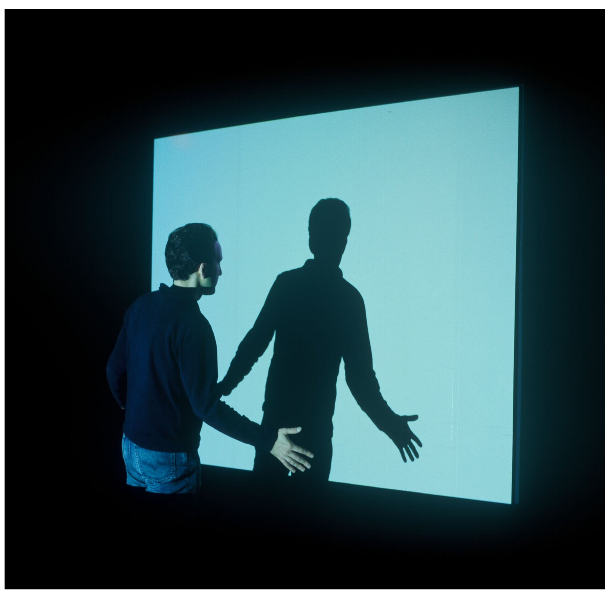
Section 4.3 Image and Inverse Image of a Set

Purpose of Section: To introduce the concept of the image and inverse image of a set. We show that unions of sets are preserved under a mapping whereas only one-to-one functions are preserved. On the other hand the inverse function preserves both unions and intersections.

Introduction

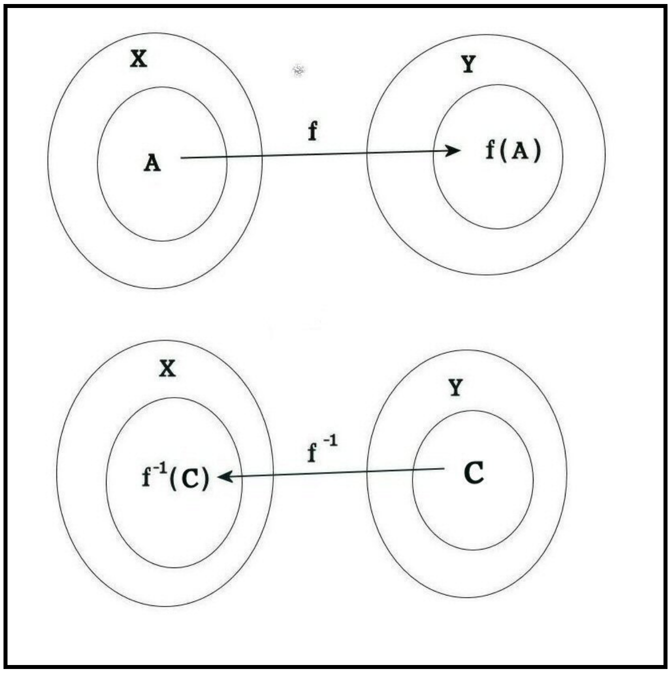
Parallel light shining on an object creates a shadow behind the object, possibly falling on a wall, which defines a mapping that sends points on the object where light strikes to its "shadow" point on a wall; i.e. the point of continuation of the ray of light on the wall if the ray could pass through the object. Although we can think of this map as a function sending points on the object to points on the wall, a more useful way is to think of it as a mapping between sets; i.e. a set of points on the surface of the object to the shadow set on the wall.



Often in mathematics, particularly in analysis and topology, one is interested in finding the set of image points of a function acting on a given set, which brings us to the following definition.

Definition: Let $f : X \to Y$ and $A \subseteq X$. As x moves about the set A, the set of values $f(x)$ define the **image of** A. That is

$$f(A) = \{ f(x) : x \in A \}.$$



Also if $C \subseteq Y$ we define the **inverse image of** C as the set

$$f^{-1}(C) = \{ x \in X : f(x) \in C \}.$$

Note that $f^{-1}(C)$ is a well defined set regardless of whether the function f has an inverse.

Example 1 Let $X = \{1,2,3,4\}$ and $Y = \{a,b,c\}$ and define a function f on $A = \{1,2,3\}$ by $f(1) = a,\ f(2) = a,\ f(3) = c$. Then $f(A) = \{a,c\}$. Also for example $f^{-1}(\{b,c\}) = \{3\}$. $f^{-1}(\{a,c\}) = \{1,2,3\}$, $f^{-1}(\{b\}) = \varnothing$. See Figure 1.

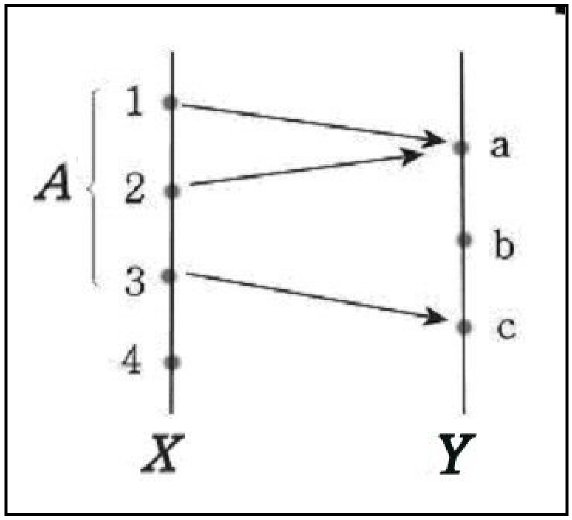
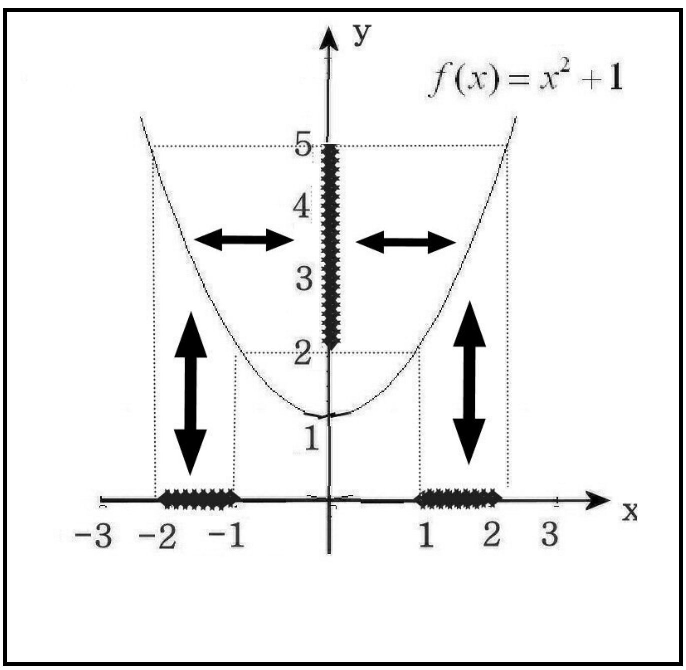


Figure 1

Example 2 Let $f : X \to Y$ be given by $f(x) = 1 + x^2$. See Figure 2.



$f^{-1}([2,5]) = [-2,-1] \cup [1,2]$

Figure 2

Then we have

a) $f(\{-1,1\}) = \{2\}$ since both $f(-1) = f(1) = 2$.

b) $f([-2,2]) = [1,5]$

c) $f([-2,3]) = [1,10]$

d) $f^{-1}(\{1,5,10\}) = \{-3,-2,0,2,3\}$

e) $f^{-1}([0,1]) = \{0\}$

f) $f^{-1}([2,5]) = [-2,-1] \cup [1,2]$

Margin Note: It is often important to know if certain properties of sets are preserved under certain types of mappings. For instance if X is a connected set and f a continuous function, then $f(X)$ is also connected.

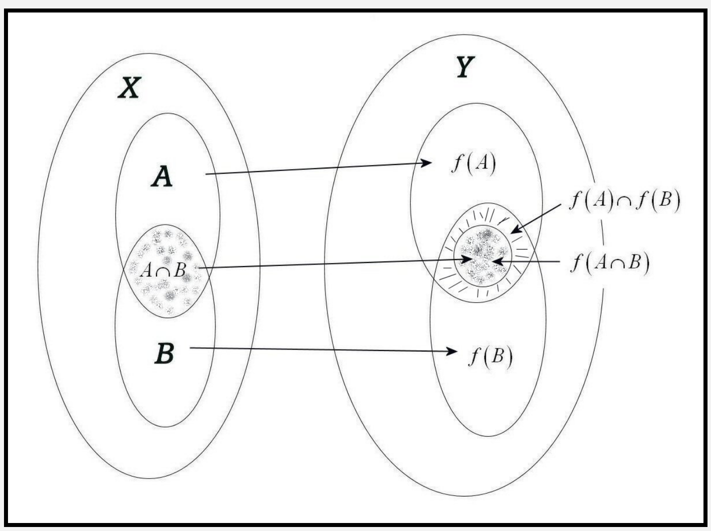
Images of Intersections and Unions

The following theorem gives an important relation of the image of the intersection of two sets.

Theorem 1

If $f : X \to Y$ and $A \subseteq X$, $B \subseteq X$, then the images of intersections satisfy

$$f(A \cap B) \subseteq f(A) \cap f(B)$$



Proof: Let $y \in f(A \cap B)$. Hence, there exists an $x \in A \cap B$ such that $f(x) = y$. Hence $x \in A$ and $x \in B$. But

$$x \in A \Rightarrow y = f(x) \in f(A)$$
$$x \in B \Rightarrow y = f(x) \in f(B)$$

Hence $y = f(x) \in f(A) \cap f(B)$.

Let us now try to show the converse; that is $f(A) \cap f(B) \subseteq f(A \cap B)$. Letting $y \in f(A) \cap f(B)$ we have $y \in f(A)$ and $y \in f(B)$. Hence

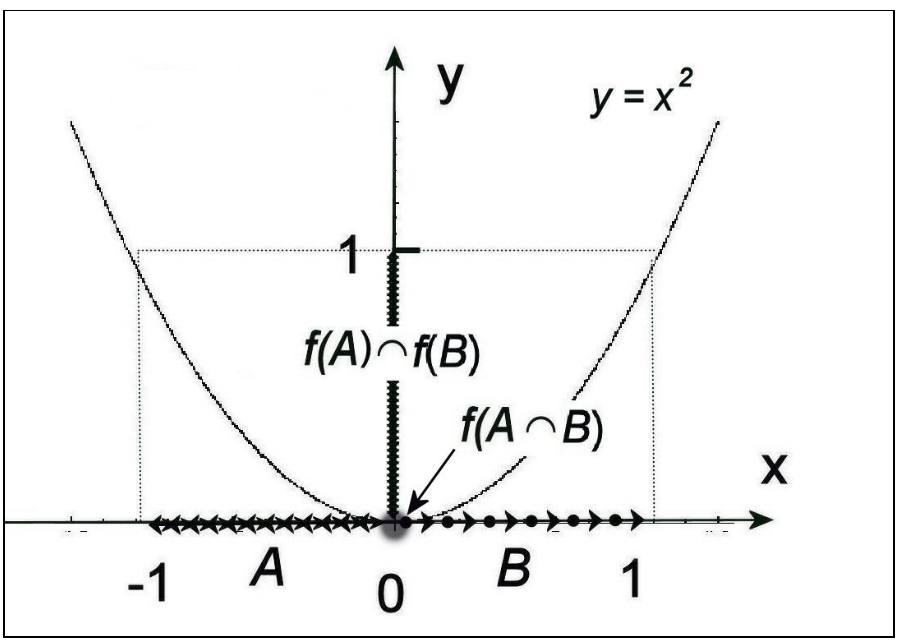
$$x_1 \in A \Rightarrow f(x_1) = y$$
$$x_2 \in A \Rightarrow f(x_2) = y$$

from which we conclude $f(x_1) = f(x_2)$ which is only true if the function f is one-to-one. Hence, we suspect the conclusion is false in general, and thus we look for a counterexample. That is is a non $1-1$ function $f\ X \to Y$ and subsets $A, B \subseteq X$ and a point $y \in f(A) \cap f(B)$ that is *not* in $f(A \cap B)$.

To find a counterexample let $X = Y = \mathbb{R}$ and $A = [-1, 0]$, $B = [0, 1]$. Then if f is defined by $f(x) = x^2$, we have

$$A \cap B = \{0\} \Rightarrow f(A \cap B) = \{0\}$$
$$f(A) = f(B) = [0,1] \Rightarrow f(A) \cap f(B) = [0,1]$$

Hence $f(A) \cap f(B) \not\subset f(A \cap B)$. See Figure 3. ∎



Counterexample to show $f(A) \cap f(B) \not\subset f(A \cap B)$

Figure 3

In general, intersections of sets are not preserved under the image of a function. However, the following theorem shows intersections are preserved.

Theorem 2 Let $f : X \to Y$ be a *one-to-one* function. If $A \subseteq X$, $B \subseteq X$, then intersection is preserved under mappings:

$$f(A \cap B) = f(A) \cap f(B).$$

Proof: It suffices to show $f(A) \cap f(B) \subseteq f(A \cap B)$. Letting $y \in f(A) \cap f(B)$, we have $y \in f(A)$ and $y \in f(B)$. Hence

$$\left| \begin{array}{l} \exists x_1 \in A \text{ such that } f(x_1) = y \\ \exists x_2 \in B \text{ such that } f(x_2) = y \end{array} \right. \Rightarrow f(x_1) = f(x_2)$$

But f is assumed 1-1 so we conclude $x_1 = x_2$ and so

$$x_1 \in A \text{ and } x_1 \in B \text{ and thus } x_1 \in A \cap B$$

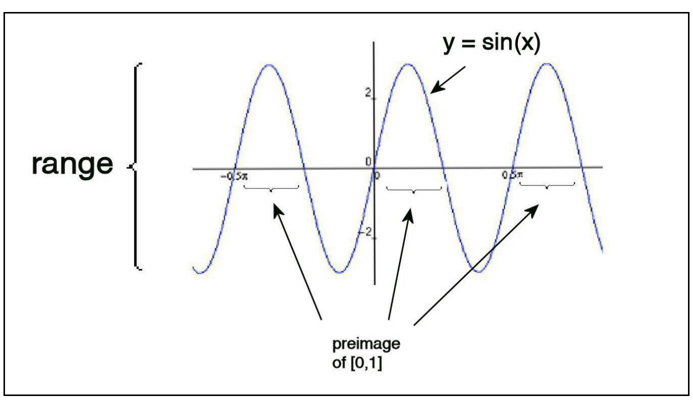
Hence $y = f(x_1) \in f(A \cap B)$ which proves theorem. ∎

Example 3 For the function $f : X \to Y$ defined by $f : x \to \sin x$, identify the domain, range, image of f, and the preimage of $[0,1]$. See Figure 4

Solution Both the domain and codomain of f are the real numbers, the range of f is $[-1,1]$, and the preimage of $[0,1]$ is

$$f^{-1}[0,1] = \{x : x \in [2n\pi, (2n+1)\pi],\ n \in \mathbb{Z}\}.$$

Note that the image of f is only a subset of the codomain.



Sine graph
Figure 4

Although we have seen that intersections are not always preserved under the action of a function unless the function is one-to-one, unions of sets are always preserved .

Theorem 3 Let $f : X \to Y$. If $A \subseteq X$, $B \subseteq X$, then unions are preserved. That is

$$f(A \cup B) = f(A) \cup f(B).$$

Proof: We begin by showing

$$f(X \cup Y) \subseteq f(X) \cup f(Y)$$

If $y \in f(A \cup B)$ this means there exists an $x_0 \in A \cup B$ such that $y = f(x_0)$. Hence, $x_0 \in A$ or $x_0 \in B$ and so $f(x_0) \in f(A)$ or $f(x_0) \in f(B)$, and so we have $f(x_0) \in f(A) \cup f(B)$ which verifies $f(A \cup B) \subseteq f(A) \cup f(B)$. We leave the verification that $f(A) \cup f(B) \subseteq f(A \cup B)$ to the reader. See Problem 4. ∎

Although the intersection of sets is not preserved for functions, it is preserved for the inverse of a function.

Theorem 4 Let $f : X \to Y$ and $C \subseteq Y, D \subseteq Y$, then the inverse image of intersections and unions satisfies

a) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

Proof: We prove a) by the series of equivalent statements:

$$\begin{aligned} y \in f^{-1}(C \cap D) &\Leftrightarrow f(y) \in C \cap D \\ &\Leftrightarrow f(y) \in C \text{ and } f(y) \in D \\ &\Leftrightarrow y \in f^{-1}(C) \text{ and } y = f^{-1}(D) \\ &\Leftrightarrow y \in f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$ ∎

The proof of b) is left to the reader. See Problem 5.

Summary:

Given a function $f : X \to Y$ where A, B are subsets of X , and C, D are subsets of Y , we have the following properties. Notice how the inverse image always preserves unions and intersections, although not always true for the image of a function.

- $f(A \cup B) = f(A) \cup f(B)$
- $f(A \cap B) \subseteq f(A) \cap f(B)$ $\quad$ (= if f is 1-1)
- $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
- $f\left(f^{-1}(C)\right) \subseteq C$ $\quad$ (= if f is 1-1)
- $A \subseteq f^{-1}\left(f(A)\right)$ $\quad$ (= if f is 1-1)
- $A \subseteq B \Rightarrow f(A) \subseteq f(B)$
- $C \subseteq D \Rightarrow f^{-1}(C) \subseteq f^{-1}(D)$
- $f^{-1}(\sim C) = \sim f^{-1}(C)$.
- $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$
- $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$
- $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$ $\quad$ (= if f is 1-1)
- $f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f^{-1}(A_i)$
- $f^{-1}\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f^{-1}(A_i)$

Problems

5. Translate the following statements. For example $y \in f(A)$ means there exists an $x \in A$ such that $y = f(x)$.

a) $y \in f(A) \cup f(B)$
b) $y \in f(A) \cap f(B)$
c) $y \in f(A \cup B)$
d) $y \in f(A \cap B)$
e) $y \in \bigcup_{i \in I} f(A_i)$
f) $y \in \bigcap_{i \in I} f(A_i)$
g) $y \in f\left(\bigcup_{i \in I} A_i\right)$
h) $y \in f\left(\bigcap_{i \in I} A_i\right)$

6. Let $f(x) = x^2 + 2$. Find the following.

a) $f(\{-1,1,3\})$
b) $f(\varnothing)$
c) $f([0,2])$
d) $f([-1,2] \cup [3,5])$
e) $f^{-1}([-1,2])$
f) $f^{-1}([0,2])$
g) $f^{-1}([4,10])$
h) $f^{-1}([-1,3] \cup [11,18])$

7. Let $f : A - \{0\} \to B$ be defined by $f(x) = |x| + 1$. Find

a) $f([-2,-1))$
b) $f([-2,3])$
c) $f([-2,-1] \cup [2,4])$
d) $f^{-1}([0,4])$

e) $f^{-1}([-2,0])$
f) $f^{-1}(\{1,2,3\})$

8. Let $f: A \to B$ where $A=\{1,2,3,4\}, B=\{a,b,c,d\}$. If

$$f=\{(1,a),(2,a),(3,c),(4,b)\}$$

find the following.

a) $f(\{2,4\})$
b) $f(\{1,3\})$
c) $f^{-1}(\{a,d\})$
d) $f^{-1}(\{b,c\})$

9. **(Identity or Falsehood?)** Let $f: A \to B$ and let X, Y be subsets of A. Prove or disprove

$$A \subseteq B \Rightarrow f(A) \subseteq f(B).$$

10. **(Families of Sets)** Let $f: X \to Y$ and $A_i \subseteq X$, $i \in I$ is a family of subsets of X. Prove the following.

a) $f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i)$
b) $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$

11. **(Image of a Union)** Show $f(A \cup B) \subseteq f(A) \cup f(B)$.
12. **(Inverse of Union)** Show $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
13. **(Compliment Identity)** Show $f^{-1}(\sim A) = \sim f^{-1}(A)$

14. **(Composition of a Function with Its Inverse)** Show the following and give examples to show we do not have to prove equality.

a) $f\left[f^{-1}(A)\right] \subseteq A$
b) $f^{-1}\left[f(A)\right] \supseteq A$

15. Let $f:\mathbb{N}\to\mathbb{R}$ be a function defined by $f(n)=1/n$. Find

a) $f^{-1}\left(\left[\frac{1}{10},1\right]\right)$

b) $f^{-1}\left(\left[\frac{1}{100},\frac{1}{2}\right]\right)$

16. **(Classroom Puzzle)** Let A be the set of students in your Intro to Abstract Math Class and B be the natural numbers from 1 to 100. Suppose now we assign to each person in the class the age of that person. That is, if x is a student, then , where n is the age of x. If we now assign to each natural number n in $f(A)$ those students whose age is n. Under what conditions will this be a function from $f(A)$ to A.

17. **(Inverse Image of an Open Interval)** In topology, a continuous function f is defined to be a function such that the inverse image of every open set is an open set. Show that for the continuous function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2-2x$, the inverse image of the following open intervals[2] is an open interval or the union of open intervals.

a) $f^{-1}((-1,0))$
b) $f^{-1}((0,2))$
c) $f^{-1}((-2,1))$
d) $f^{-1}((2,6))$

18. **(Dirichlet's Function)** Dirichlet's function[3] $f:[0,1]\to R$ is defined by by

$$f(x)=\begin{cases}1 & x \text{ is rational}\\ 0 & x \text{ is irrational}\end{cases} \qquad 0\le x\le 1$$

Find

a) $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$

b) $f^{-1}\left(\left[0,\frac{1}{2}\right]\right)$

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[2] Open intervals and union of open intervals are special cases of open sets and the real number system is a special topological space.

[3] Sometimes called the "shotgun" function since it is full of holes

19. **Properties of Images** One can prove for a continuous function $f:\mathbb{R}\to\mathbb{R}$, the following properties hold:

a) the image $f\left([a,b]\right)$ of a closed interval $[a,b]$ is not necessarily a closed interval.

b) The image $f(A)$ of a bounded set A is not necessarily bounded.
c) The image of a closed and bounded interval is closed and bounded.
d) The inverse image $f^{-1}(a,b)$ of an open interval is an open set; either an open interval or the union of open intervals.

Give examples of a function f and domains where these properties hold.

20. Let $f:X\to Y$. Show that for $x\in X$ one has $f\left(\{x\}\right)=\{f(x)\}$.

21. **(Connected Sets)** It can be proven that the continuous image of a connected set is connected. Show that the image of the connected set $(-1,1)$ under the functions.

a) $f(x)=x^3$
b) $f(x)=e^x$
c) $f(x)=2x+1$