Section 4.1 The Function

**Purpose of Section** To introduce the concept of the function, both as a “rule” which assigns a unique value to every member of a given set, and from the relation viewpoint, which interprets functions as subsets of ordered pairs in a Cartesian product.

**Introduction**

Of all the relations in mathematics, the function relation occupies center stage. No doubt the concept of a function covers familiar territory to most readers of this book\(^1\). Normally, in most beginning mathematics texts, a function \( f : A \to B \) is defined to be a “rule” that assigns to each value \( x \in A \) a unique value \( y \in B \). This is the definition proposed by German mathematician Peter Lejeune Dirichlet (1805–1859) in the 1830s. This definition of sometimes criticized since it leaves open the question, what is the rule? When we write an algebraic formula, such as

\[
y = f(x) = \sin x
\]

where \( x \) is a real number, it is understood that the rule, which we denote by \( f \), which assigns to each real number \( x \) the real number \( y = \sin x \). We call \( f(x) \) the value of the function at \( x \). In other words, a function which assigns to any real number \( x \in \mathbb{R} \) a unique real number \( f(x) \in \mathbb{R} \) defines binary relation on \( \mathbb{R} \), which we denoted by \( f = (x, f(x)) \in \mathbb{R} \times \mathbb{R} \).

This discussion motivates the formal definition of the function due to the German mathematician Johann Dirichlet (1805–1859).

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\(^1\) A fascinating reference for functions is *Atlas for Computing Mathematical Functions* by William Jackson Thompson which gives analytical, visual and descriptive properties of over 150 special functions useful in pure and applied mathematics.
Definition  Let $A$ and $B$ be sets. A function $f$ (or mapping) from $A$ to $B$, denoted $f : A \to B$, is a rule which assigns to each element in $A$ a unique element in $B$. The set $A$ is called the domain of the function, and $B$ is called the codomain of the function. For $x \in A$ the assigned value in $B$ is called the image of $x$ under $f$ and is denoted by $f(x)$, which is read “the value of $f$ at $x$.”

It is sometimes useful to think of a function $f : A \to B$ as a ‘black box’ where a value of $x \in A$ is entered into the box, whereupon the box performs some operation on $x$, turning out a value $y = f(x) \in B$.

For a function $f$ whose domain and codomain is $A$ and $B$ respectively, its graph is the set

$$\text{graph}(f) = \{(x, y) : x \in A, y = f(x) \in B\}$$

When $A$ and $B$ are real numbers the graph can be drawn as shown in Figure 1.

A test whether a relation $f$ on $\mathbb{R}$ is a function is the vertical line test, which requires that any vertical line drawn through a point in the domain intersects the
graph in one and only one place. Figure 2 shows a relation that is a function and a relation that is not a function.

Basic Terminology

For a function $f : A \to B$ we define the following:

- **Range:** The set $f(A) = \{ f(x) : x \in A \} \subseteq B$ is called the range of $f$, denoted $\text{Rng}(A)$. The range need not be the entire codomain $B$. That is, it is not necessarily true that $f(A) = B$, only that $f(A) \subseteq B$.

- **Image:** If $S \subseteq A$, the set $f(S) = \{ f(x) : x \in S \}$ is called the image of $S$. 

Vertical Line Test

Figure 2
There are several synonyms used for the word “function.” The words *mapping* (or *map*), *transformation*, and *operator* are often used depending on the domain and codomain of the function. We are so accustomed to functions whose domain and codomain are the real numbers, we don’t think of functions with different types of domains as functions.

**Example 1 (Functions)**

The following are functions with different domains and codomains.

a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{1-x^2}$ has domain $[-1,1]$ and codomain $\mathbb{R}$. The range of $f$ is $[0,1] \subseteq \mathbb{R}$.

b) The function $f : \mathbb{N} \to \mathbb{R}$ defined by the sequence $f(n) = \sin n$, $n = 1, 2, \ldots$ has domain the natural numbers $\mathbb{N}$ and codomain the real numbers $\mathbb{R}$. The range is the set

$$\text{range}(f) = \{\sin n, n = 1, 2, \ldots\}$$

c) The function $f : \mathbb{R} \leftrightarrow \mathbb{R}^3$ defined by

$$f(t) = (t, \sin t, \cos t)$$

is a curve (helix) in three-dimensional space. The domain is the set of real numbers and the codomain is three dimensional space. The image at $t \in \mathbb{R}$ is the helix $(t, \sin t, \cos t) \in \mathbb{R}^3$.

d) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$y = f(x_1, x_2) = 3x_1 + 2x_2$$
defines a function (transformation) whose domain is $\mathbb{R}^2$ and codomain is the real numbers. The image of the point $(x_1, x_2) \in \mathbb{R}^2$ is $3x_1 + 2x_2 \in \mathbb{R}$.

e) The transformation $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}$$

has a domain $\mathbb{R}^2$ and codomain $\mathbb{R}^2$. The image of the point $(x, y) \in \mathbb{R}^2$ is the point $(u, v) \in \mathbb{R}^2$.

f) The functional $T : y \rightarrow T(y)$ defined by the integral

$$T(y) = \int_a^b \sqrt{1 + \left( \frac{y'(x)}{2g(y(x)} \right)^2} \, dx$$

gives the time of descent $T(y)$ for a particle to slide without friction under the force of gravity from a higher point $y(a)$ to lower point $y(b)$ down the curve defined by $y(x)$. The domain of $T$ is the class of decreasing differentiable curves $C^1(a, b)$ and the codomain is the positive real numbers.

A Short History of the Function

The interest in things related is rooted in antiquity, going all the way back to Greek times when geometers studied relations between geometric objects. Fast forward two thousand years, the co-inventors of calculus, Newton and Leibniz wrote about the interdependence of moving quantities. One of the early non-geometric definition of a function was due to the Swiss mathematician Leonard Euler (1707–1783), who defined a function as

Quantities dependent on others, such that as the second changes, so does the first, are said to be functions.

Euler and other leading mathematicians of the times thought of functions in terms of equations, such as $y = x^3$. An expression of the form
was not considered a function since it simply was a “rule” for assigning values to \( x \). However, the \textit{equation} requirement for a function ran into a paradox when Swiss mathematician Daniel Bernoulli and French mathematician Jean le Rond d’Alembert solved the \textit{vibrating string problem}, and got the same vibrations for different equations, prompting mathematicians to think about the interpretation of a function. In 1837, German mathematician Peter Lejeune Dirichlet (1805–1859) proposed the modern definition of a function that we know today:

\[
A \text{ variable quantity } y \text{ is said to be a function of a variable quantity } x, \text{ if to each value of } x \text{ there corresponds a uniquely determined value of the quantity } y.
\]

**Relation Definition of a Function**

In addition to interpreting a function as a rule, (ala Dirichlet), it is also possible to think of a function is as a relation, a subject we are well familiar, having studied order and equivalence relations in the previous chapter. The relation definition of a function is as follows.

**Relation Definition:** A function \( f \) from \( A \) to \( B \) is a relation from \( A \) to \( B \) (i.e. \( f \subseteq A \times B \)) that satisfies

\[
[(x, y) \in f \land (x, z) \in f] \implies y = z
\]

for \( \forall x \in A \) and \( \forall y, z \in B \)

**Note:** The above “relation” definition of a function is generally called the \textit{graph} of a function. Whether one accepts the Dirichlet “rule” definition of a function or the “relation” definition is irrelevant. They are simply different ways to think of the same thing\(^2\).

**Example 2 Relation Definition of \( f \)**

\(^2\)From a purely logical point of view, there is some merit in the relational definition of a function inasmuch it is defined in terms of sets, and sets are based on logical axioms. Nowadays, however the Dirichlet definition is the more common way most mathematicians think of functions.
Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5\}$. Which of the following relations on $A \times B$ are functions?

a) $R_1 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$  
b) $R_2 = \{(1,2),(2,1),(3,5),(1,3),(5,5)\}$  
c) $R_3 = \{(1,1),(2,1),(3,1),(4,1),(5,1)\}$  
d) $R_4 = \{(1,1),(2,2),(3,1),(4,1),(3,4)\}$

**Solution:**

a) is a function (also an order and equivalence relation) 
b) is not a function since 1 maps into two values, 2 and 3  
c) is a function (constant function)  
d) is not a function since 3 maps into more than one value, 1 and 4.

**Example 3 (Order, Equivalence, and Function)**

Figure 3 shows three relations on the set $A = \{1, 2, 3, 4\}$. One relation is an order relation, one an equivalence relation, and the other a function relation. Which is which?

![Figure 3](image)

**Solution**

- The relation in Figure 3a) is the only relation of the three that is reflexive, symmetric and transitive, hence is the equivalence relation. It is not a function so it would be incorrect to write $f : A \to A$.  

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3 Note that the equivalence classes are $\{1\}, \{2, 3\}, \{3\}$. 

The relation in Figure 3b) is reflexive, antisymmetric and transitive, so it is the order relation\(^4\). In fact it is the relation \(x \leq y\).

Figure 3c) is the only relation that has exactly one \(y\) value for each \(x\) so it is the function relation\(^5\).

**Margin Note:** Two ways to represent a function.

Dirichlet rule form: \(f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x + \sin x\)

Relation form: \(f = \{(x, y) : y = x + \sin x\}\)

### Composition of Functions

In Section 3.1 we introduced the composition of a relation and since functions are relations, functions have compositions as well. However, we now define a function as a rule \(f : A \rightarrow B\) rather than as a subset \(f \in A \times B\) so the definition takes a different (but equivalent) form.

### Composition of Two Functions

Let \(g : A \rightarrow B\) and \(f : B \rightarrow C\) be functions. The composition (or composite) of \(f\) and \(g\), denoted by \(f \circ g : A \rightarrow C\) is defined by \((f \circ g)(x) = f(g(x))\) for all \(x\) in the domain of \(g\) and \(g(x)\) in the domain of \(f\).

![Composition of Two Functions Diagram]

**Example 4 (Composition of Real–Valued Functions)**

Find the compositions \(f \circ g\) and \(g \circ f\) of the functions \(f(x) = x^2\) and \(g(x) = \sqrt{x}\).

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\(^4\) In fact the order relation is \(x \leq y\)

\(^5\) If we named the function \(f\), then we would have \(f(1) = 2, f(2) = 3, f(3) = 2, f(4) = 4\).
Solution

\[(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x \quad x \in [0, \infty)\]

\[(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x| \quad x \in \mathbb{R}\]

Note that not only that \(f \circ g \neq g \circ f\) but that their domains are not the same.

We can also take the composition of operators: functions whose domain and range are families of functions.

**Margin Note:** We can interpret the composition \((f \circ g)(x) = f(g(x))\) of two functions in terms of black boxes. We put an \(x\) in the first black box \(g\) and get the output \(g(x)\), where we then put \(g(x)\) in the second black box \(f\) and we get the output \(y = (f \circ g)(x)\).

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**Example 5 (Differential Operators)** Define operators \(D\) and \(M\) by

\[D(f) = \frac{df}{dx} \quad M(f) = xf(x)\]

where we assume the domain and codomain of the operators are chosen so all expressions make mathematical sense. The composition of these operators is

\[(D \circ M)(f) = D(M(f)) = D(xf) = x \frac{df}{dx} + f\]

\[(M \circ D)(f) = M(D(f)) = M\left(x \frac{df}{dx}\right) = x \frac{df}{dx}\]
Problems

1. **(Testing Relations)** Which of the following relations are functions? For functions what is the domain and range of the function?
   
   a) \( R = \{(1,3),(3,4),(4,1),(2,1)\} \)
   
   b) \( R = \{(1,3),(1,4),(1,2),(3,1)\} \)
   
   c) \( R = \{(1,3),(3,4),(1,1)\} \)
   
   d) \( R = \{(1,2),(2,2),(3,2),(2,3)\} \)

2. **(Graphing Relations and Functions)** Graph each of the following relations on \( \mathbb{R} \) and tell which relations are functions.
   
   a) \( \{(x,y): x = 1\} \)
   
   b) \( \{(x,y): y = 1\} \)
   
   c) \( \{(x,y): y = x^2\} \)
   
   d) \( \{(x,y): x = y^2\} \)
   
   e) \( \{(x,y): x = \sin y\} \)
   
   f) \( \{(x,y): |x|+|y|=1\} \)

3. Find \( f \circ g \) and \( g \circ f \) and their domains for the given functions \( f, g \) where the domains of the functions are assumed to be all real values for which the function is well-defined.
   
   a) \( f(x) = x^2 + 1, \ g(x) = |x| \)
   
   b) \( f(x) = e^x, \ g(x) = \ln x \)
   
   c) \( f(x) = \frac{1}{x^2+1}, \ g(x) = x^2 \)
   
   d) \( f(x) = |x|, \ g(x) = |x| \)
   
   e) \( f(x) = x^2 + 1, \ g(x) = x \)
   
   f) \( f(x) = \sqrt{1-x^2}, \ g(x) = \sqrt{x^2 - 1} \)
   
   g) \( f(x) = 2\sin x, \ g(x) = 1 \)
   
   h) \( f(x) = 2, \ g(x) = 3 \)

4. **(Iterates of a Function)** If \( f(x) = \cos x \) and starting at an arbitrary initial value \( x_0 = 1 \) compute the iterates \( x_n = f(x_{n-1}), n = 1, 2, \ldots \). Draw a picture and
then prove why the sequence \( \{x_n\} \) of values gets closer and closer to 0.739085.

5. **(Classroom Function)** Let \( A \) be the set of students in your Intro to Abstract Math Class and \( B \) be the natural numbers from 1 to 100.

a) Suppose we assign to each student in the class the age of that student. That is, if \( x \) is a student in the class, then \( f(x) \) is the age of \( x \). Is this a function from \( A \) to \( B \)?

b) Suppose we assign to each natural number \( n \in B \) all students in \( A \) whose age is \( n \). Is this a function from \( B \) to \( A \)?

6. **(Finding Compositions)** Give examples of \( f, g \) that satisfy the following compositions.

   a) \( (f \circ g)(x) = x - 2 \)
   b) \( (f \circ g)(x) = e^{x^2} \)
   c) \( (f \circ g)(x) = x - 2 \)
   d) \( (f \circ g)(x) = \sqrt{x^2 + 1} \)

7. **(More Compositions)** Given functions \( f, g \) each with domain and codomain \( A = \{1, 2, 3, 4\} \) as illustrated in Figure 4, find the following.

![Figure 4](image)

   a) \( f \circ g \)
   b) \( g \circ f \)
   c) \( f \circ f \)
   d) \( g \circ g \)
8. **(Strictly Increasing)** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strictly increasing if $a < b \Rightarrow f(a) < f(b)$. If $f, g$ are both strictly increasing real-valued functions defined on the real line, then is $f \circ g$ strictly increasing? Either prove it or find a counterexample.

9. **(Hmmmmmmmmmm)** If $f, g$ are both differentiable real-valued functions defined on the real line, is their composition $f \circ g$ differentiable?

10. **(Graphing a Composition)** Draw an arbitrary graph of two real-valued functions of a real variable $f, g$ and select an arbitrary real number $x$. Illustrate on the graph the location of $(f \circ g)(x)$.

11. **(Composition)** Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 + x^3, \quad g(x) = \sqrt[3]{x}$. Show that $f \circ g \neq g \circ f$. Are the domains of the two compositions the same?

12. **(Compositions)** Find the composition $f \circ g$ of the functions $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = (t, t^2, t^3), \quad g(t) = \sin t$. What is the domain of the composition?

13. **(Composition of Operators)** Define operators $L_1(f) = xf(x) + 1, \quad L_2(f) = x^2 \frac{df}{dx}$ on domains sufficient for all operators to exist. Find

   a) $L_1 \circ L_2$

   b) $L_2 \circ L_1$

14. **(Characteristic Function)** The characteristic function of a set $A$ is defined as $f_A = \{(x, 1) : x \in A\} \cup \{(x, 0) : x \notin A\}$. Draw the graph of the characteristic function of $A = [1, 2] \cup [3, 4]$.

15. **(A Function or Not a Function)** Given the set of points $(x, y)$ in the plane that satisfy the equation $|x| + |y| = 1$

   a) Draw the points that satisfy the equation. (Hint: Draw each quadrant separately.)

   b) Does the relation define a function?
16. **(Composition of Onto Functions)** Prove that if $g$ maps $X$ onto $Y$, and $f$ maps $Y$ onto $Z$, then the composition $f \circ g$ maps $X$ onto $Z$.

17. **(Composition of 1-1 Functions)** Prove that if $g$ is a 1-1 mapping from $X$ to $Y$, and $f$ is a 1-1 mapping from $Y$ to $Z$, then the composition $f \circ g$ is a 1-1 mapping from $X$ to $Z$.

18. **(Pathological Function)** The Weierstrass function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$$

(where $0 < a < 1$, $b$ is any odd positive integer, $ab > 1 + \frac{3\pi}{2}$) is a function that has the non-intuitive property that it is continuous everywhere, but differentiable nowhere. It was published by Karl Weierstrass (the “father” of rigor in analysis) to challenge the belief at the time that every continuous function was differentiable except on a set of isolated points. Use a computer algebra system to plot the first What is your intuitive interpretation of such a function and what would it look like?