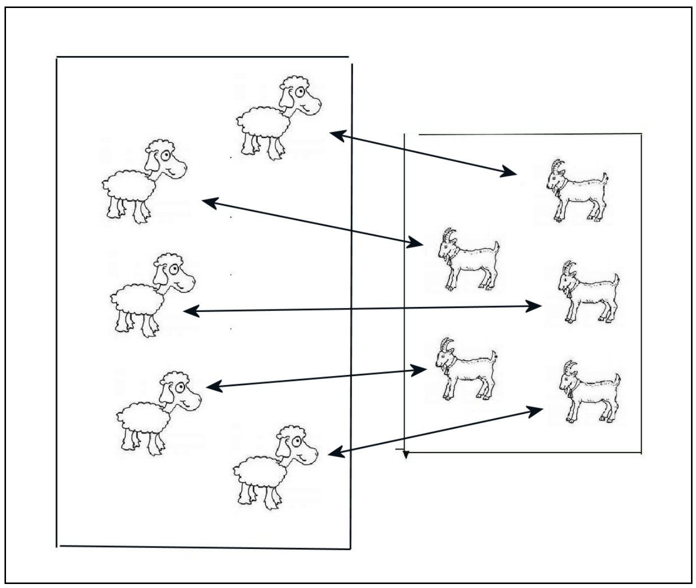
Section 2.4 Countable Infinity

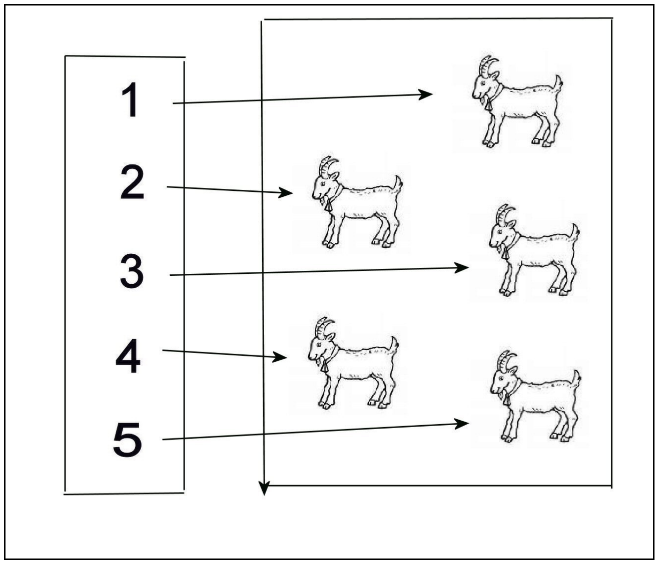
> **Purpose of Section:** To introduce the concept of **equivalence** of sets and the **cardinality** of a set. We present Cantor's proof that the rational and natural numbers have the same cardinality.

Introduction

No one knows exactly when people first started counting, but a good guess might be when people started accumulating possessions. Long before number systems were invented, two people could determine if they had the same number of goats and sheep by simply placing them in a one-to-one correspondence with each other.



Or a person could have a stone for each goat, hence obtaining a one-to-one correspondence between the stones and the goats. Today we no longer need physical stones since we have *symbolic* ones in the form of 1, 2, $\cdots$. To determine the number of goats we simply "count," 1,2, $\cdots$ and collect our rocks $R = \{1,2,3,4,5\}$ in our mind.



So how do we compare the size of two sets? Clearly $A=\{1,3\}$ contains two elements and $B=\{a,b,c\}$ has three elements, so we say B is "larger" than A. These ideas are fine for finite sets, but how do we compare the "size" of infinite sets, like $\mathbb{N}$ **and** $\mathbb{R}$? Throughout the history of mathematics, the subject of infinity has been mostly taboo, more apt to be part of a discussion on religion. The Greek philosopher Aristotle (*circa* 384-322 b.c.), one of the first mathematicians to think seriously about the subject, felt there were two kinds of infinity, the *potential* and *actual*. He said the natural numbers 1, 2, 3, ⋯ are *potentially* infinite since the numbers never stop, but they were not a completed entity. Philosopher and theologian Thomas Aquinas (1225-1275) argued that with the exception of God nothing was actual infinite, only potential.

In the 1600s the Italian astronomer Galileo made an interesting observation concerning the perfect squares 1, 4, 9, 16, 25, ⋯ Since they constitute a subset of the natural numbers, he argued there should be "fewer" of them than the natural numbers, and Figure 1 would seem to bear this out.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1			4					9							16

More natural numbers than perfect squares
Figure 1

However, Galileo also observed that if you *line up* the perfect squares as in Figure 2, it appears that both sets have the *same* number of members.

1	2	3	4	5	6	7	8	9	10	11	$\cdots$	n	$\cdots$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$	$\updownarrow$
1	4	9	16	25	36	49	64	81	100	121	$\cdots$	n^2	$\cdots$

Equal number of perfect squares as natural numbers.
Figure 2

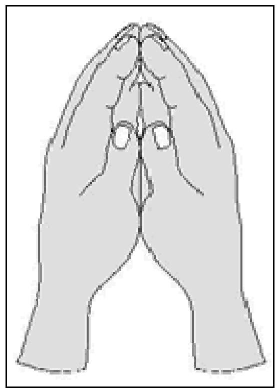
His argument was that for every perfect square n^2, there is exactly one natural number n, and conversely, for every natural number n there is exactly one square n^2. Galileo came to the conclusion that the concepts of "less than," "equal", and "greater than" applied only to finite sets and not infinite ones.

It was the ground-breaking work of German mathematician, Georg Cantor (1845-1918), whose seminal insights transformed the thinking about "potential versus actual infinities" with his investigations into the foundations of the real numbers leading to sets of different sizes of infinite sets, and topics which had previously been "off limits" in mathematics. Many mathematicians resisted Cantor's ideas but by the time of Cantor's death in 1918 mathematicians recognized the importance of his ideas.

As many seminal insights, Cantor's theory of infinite sets rests upon a very simple principle. Suppose you are unable to count but would like to determine whether you have the same number of fingers on each of your two hands.

Georg Cantor (1845-1918)

Now assume you never learned arithmetic in grade school and are unable to count. But this doesn't stop you since you do something more basic. You simply place the thumb of one hand against the thumb of your other hand, then place your index finger of one hand against your index finger of your other hand, and do the same for the remaining fingers. When you are finished your fingers are matched up in a *one-to-one correspondence* : every finger on each hand has a kindred-soul on the other. You may not know how many fingers you have, but you know both hands have the same number.



This strategy many not seem like a reasonable way of doing things for small (finite) sets, but what if you had an infinite set, like the natural numbers, rational numbers, or real numbers? In these cases it doesn't matter that you can't count. No one else can either, no one can count that high. Cantor's inspiration was that, even if we can't "count" infinite sets, we might be able to tell if two different infinite sets have the *same* number of elements by simply applying the procedure we used to determine if we had the same number of fingers on each hand. We see if we can put the elements of the sets in a one-to-one correspondence with each other. This leads us to the definition of the equivalence of sets.

Equivalent Sets

The determination of whether two sets have the "same number" of elements depends on whether the elements of the sets can be "paired-off" in a one-to-one fashion.

Definition Two sets A and B are **equivalent**, denoted $A \approx B$, if and only if there is a **one-to-one** and **onto** function $f: A \to B$. A function $f: A \to B$ is called one-to-one if $a, b \in A$, $a \neq b \Rightarrow f(a) \neq f(b)$ or equivalently $f(a) = f(b) \Rightarrow a = b$. A function $f: A \to B$ is onto B if for all $y \in B, \exists x \in A$ such that $f(x) = y$. If a function is both one-to-one and onto then the sets A and B can be placed in a **one-to-one correspondence** (also called a **bijection**). Equivalent sets are said to have the same **cardinality** or have the 'same number of elements'.

Example 1 The sets $A = \{a, b, c\}$ and $B = \{9, 25, 30\}$ are equivalent since we can find a bijection $f(a) = 9, f(b) = 25, f(c) = 30$ from one set to the other. (We could have just as well gone backwards.)

Example 2 Show there are the same number of perfect squares as there are natural numbers.

Proof:

Letting $\mathbb{N} = \{1, 2, 3, ...\}$ be the natural numbers and $S = \{n^2 : n \in \mathbb{N}\}$ the perfect squares of the natural numbers, we must find a one-to-one correspondence between S and $\mathbb{N}$. To do this consider the function $f: \mathbb{N} \to S$ defined by $f(n) = n^2$. Clearly for each $n \in \mathbb{N}$ we have exactly one image n^2 and so f is a function. To show f is one-to-one, let $f(u) = f(v)$, or $u^2 = v^2$, and since u, v are positive, we have $u = v$. Hence f is one-to-one. To show that f is onto, we let $y \in S$, but since y is the square of a natural number, we have $\sqrt{y} \in \mathbb{N}$ which proves that f is onto. Hence, f is a one-to-one correspondence between S and $\mathbb{N}$ and so we have proven $\mathbb{N} \approx S$.

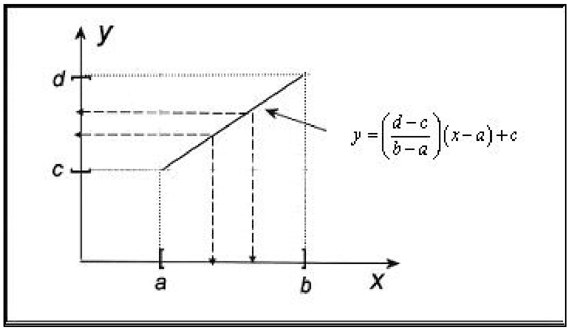
Note: It would seem that the natural numbers 1,2,3,⋯ should be "larger" than the even numbers 0,2,4,⋯ since the even numbers are only "half" the natural numbers. But what do we mean by "half" of infinity? Our experience with finite numbers must be abandoned when working with infinite sets.

Example 3 Let a, b, c, d be real numbers with $a < b, c < d$. The interval $[a, b] = \{x : a \le x \le b\}$ is equivalent to the interval $[c, d] = \{x : c \le x \le d\}$.

Proof: The function

$$y = \left(\frac{d-c}{b-a}\right)(x-a) + c$$

is a one-to-one correspondence between the points in the interval $[a,b]$ and points in the interval $[c,d]$. Figure 3 shows a visual representation of this bijection.



Equivalence of Two Intervals of Real Numbers
Figure 3

Jumping Ahead: In Chapter 3 we will see that the "relation" of two sets being equivalent is an **equivalence relation**, which is a special kind of relation (other relations being "=","<") which "partitions" families of sets into disjoint **equivalent classes.** For the equivalence relation $A \approx B$ between sets studied in this section, we associate a cardinal number, like 1, 2, 3, ⋯ to each equivalence class. One equivalence class would be called "1", another "2" and so on. The equivalence class we call "5" would consist of all sets that have "five" elements in them.

The following example shows it is not always necessary to demonstrate an algebraic form of the bijection f .

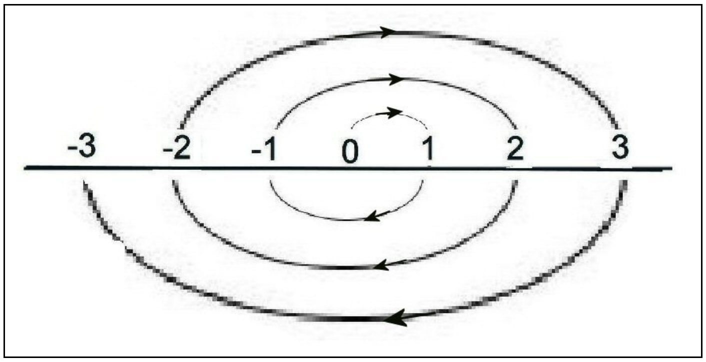
Example 4

Prove there are just as many natural numbers $N=\{1,2,3,...\}$ as there are integers $\mathbb{Z}=\{0,\pm1,\pm2,...\}$. That is $\mathbb{Z}\approx\mathbb{N}$.

Solution

The diagram in Figure 4 illustrates figuratively a one-to-one correspondence between the natural numbers $\mathbb{N}$ and the integers $\mathbb{Z}$, thus proving the natural numbers and integers have the same cardinality. The correspondence is

$$\mathbb{N}\approx\mathbb{Z}: 1\leftrightarrow 0,\ 2\leftrightarrow 1,\ 3\leftrightarrow -1,\ 4\leftrightarrow 2,\ 5\leftrightarrow -2,\ 6\leftrightarrow 3,\ ...$$



Equivalence of the Natural Numbers and the Integers
Figure 4

In case one does not prefer the "visual" correspondence as shown in Figure 4, one could find the actual bijection $f:\mathbb{N}\to\mathbb{Z}$. A bijection $f:\mathbb{N}\to\mathbb{Z}$ is

$$f(n)=\begin{cases}\dfrac{n}{2}, & n=0,2,4,\ldots\\ -\left(\dfrac{n-1}{2}\right), & n=1,3,5,\ldots\end{cases}$$

To prove f is a bijection, we must show it is one-to-one and onto. We leave this proof to the reader. (See Problem 10).

Finite and Countable Infinite Sets

Definition

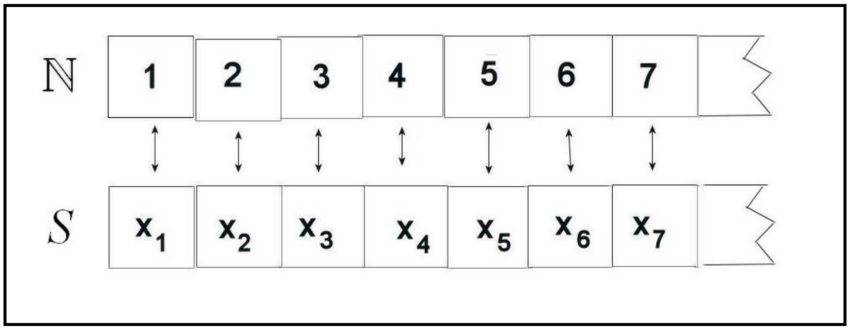
- A set A is **finite** if and only if $A=\varnothing$ or if A is equivalent to a set of the form $\mathbb{N}_n=\{1,2,\ldots,n\}$ for some natural number n. If A is equivalent to $\mathbb{N}_n$, the set A has **cardinality** n and we denote this as $|A|=n$.

- A set A is **infinite** if it is not finite.
- The **empty set** has cardinality zero, i.e. $|\varnothing|=0$.

- A set that is either finite or can be put in a one-to-one correspondence with the natural numbers is called **countable.** A set can be put in a one-to-one correspondence with the natural numbers it is called **countably infinite.** If a cardinal number is not finite, it is called **transfinite.** If a set is not countable it is called **uncountable.**

- A set that can be placed in a one-to-one correspondence with the natural numbers is an infinite set whose cardinality is called **aleph null**[1], and denoted by $\aleph_0$.

Sets of cardinality $\aleph_0$ are those sets which can be "counted" or arranged in a sequence $S = (x_1, x_2, ...)$.



Example 4 (Finite and Infinite Cardinalities)

- $A = \{\text{states in the U.S.}\} \Rightarrow |A| = 50$
- $A = \{\text{residents of Texas}\} \Rightarrow |A| > 500$
- $A = \{x \in \mathbb{R} : x^2 + 1 = 0\} \Rightarrow |A| = 0$
- $A = \{2,4,6,...\} \Rightarrow |A| = \aleph_0$
- $A = \{x \in \mathbb{R} : \sin x = 0\} \Rightarrow |A| = \aleph_0$

Note: It is the ability to "list" sets as a first, second, third, etc that characterizes countable sets.

The relation $\approx$ tells us when two sets are the same size, but we also want to know when one set is smaller or larger than another set. The following definition makes that precise.

Definition: Ordering Cardinalities Given two sets A, B, we say the cardinality of A is **less than or equal to** the cardinality of B, denoted $|A| \le |B|$, if and only if there exists a **one-to-one** function $f : A \to B$ (i.e. a one-to-one correspondence between A and a *subset* of B). If $|A| \le |B|$ but they do *not* have the same cardinality, we say the cardinality of A is **strictly less** than the cardinality of B, and denote this by $|A| < |B|$.

[1] $\aleph$ is the first letter in the Hebrew alphabet and pronounced Aleph. . $\aleph_0$ is pronounced Aleph-null or Aleph-naught, and denotes the smallest infinity.

Cardinal and Ordinal Numbers: Numbers are used in two different ways. Numbers can denote "how many" and "which one." For example, the number 3 is called a **cardinal number** when we say the "three little pigs," but when we say "the third little pig built his house out of bricks," the number three (or third) is an **ordinal number**. The sequence 1,2,3,⋯ is a sequence of cardinal numbers, the sequence first, second, third, ⋯ is a sequence of ordinal numbers.

Theorem 1 ($\aleph_0$ is the Smallest Infinity)

No infinite set has a *smaller* cardinality than the natural numbers.

Proof

Let S be an arbitrary infinite set (denumerable or uncountable). It is clear we can take away one of its members, say s_1 without emptying S. We can then take out another member s_2 without emptying S. Continuing this process, we can take out a denumerable sequence $\{s_1, s_2, ...\}$ without emptying S. This says that every infinite set contains a denumerable proper subset, which means the cardinality of a denumerable set can not be *greater* than the cardinality of any infinite set. Hence, $\aleph_0$ it is the smallest transfinite number.

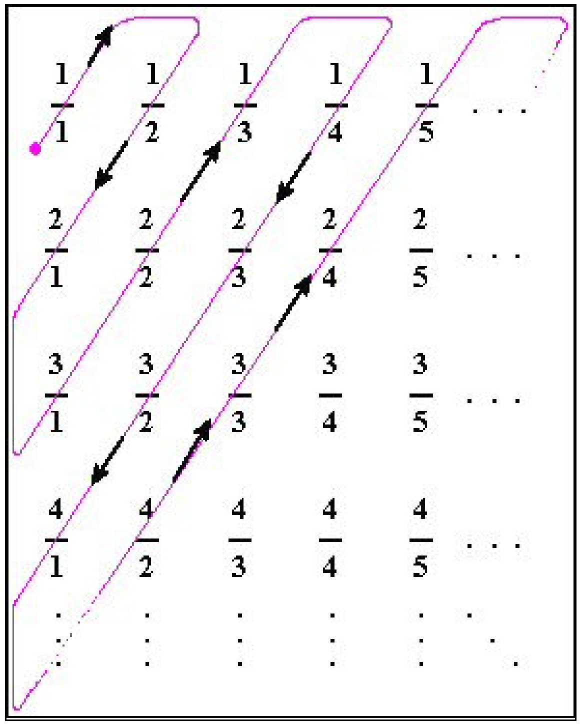
Note: The symbol "∞", the reader is well aware of from calculus, does *not* meant to stand for an infinite set. The phrase $x \to \infty$ simply refers to the fact that the variable x grows without bound.

One of the fascinating properties of infinite sets is how one set that seems so much larger than another is actually the same size or even smaller! Cantor wondered about the relative size of the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ and the rational numbers $\mathbb{Q}^+ = \{p/q : p, q \in \mathbb{N}, q \neq 0\}$. Certainly there must be more rational numbers than natural numbers; after all it can be proven that between any two real numbers, say 1 and 1.000000001, there are an infinite number of rational numbers! So what is the answer?

Cardinality of the Rational Numbers

Offhand most people would say there are "more" rational numbers than integers, but if they did they would be wrong using, at least using Cantor's system of measure. Cantor found an ingenious match up[2] between the integers and the rational numbers which is illustrated in Figure 4.

[2] A bijection between two sets need not be an equation; but simply a demonstration that it *is* a one-to-one correspondence between sets. Often this correspondence is done with a visual diagram.



One-to-one Correspondence between the
Natural Numbers and the Rational Numbers.
Figure 4

Observe that every possible (positive) fraction $p/q,\ q \neq 0$ is listed in the array in Figure 4 if you continue indefinitely downwards and to the right. Some fraction are duplications, such as $2/2 = 3/3$ and $1/3 = 3/9$ but that is ok for our purposes. Cantor now begins the one-to-one correspondence between the natural numbers and the rational numbers by counting "1" at the point 1/1 in the array, then "2" at 1/2, and so on, moving along the indicted path and skipping over the duplicates. This yields the one-to-one correspondence

1	2	3	4	5	6	7	⋯
↕	↕	↕	↕	↕	↕	↕	↕
1/1	1/2	2/1	3/1	1/3	1/4	3/2	⋯

From this Cantor concluded that the rational and natural numbers have the *same* cardinality, the cardinality $\aleph_0$ of the natural numbers. Like we said, things get strange in Cantor's world of infinity.

Problems

1. Which of the following sets are finite? If possible find the cardinality of the sets

   a) the stars in the Milky Way
   b) the atoms in a grain of sand
   c) the solutions of $x^7 + 5x^5 + x + 1 = 0$
   d) the round trip paths around the U.S. visiting each state capital exactly once

   e) five card hands dealt from a deck of 52 cards
   f) prime numbers greater than 10^{10}

   g) points (m,n) in the plane where the coordinates m,n are integers

   h) $\{n \in \mathbb{N} : n^2 \text{ is odd}\}$
   i) $\{n \in \mathbb{N} : n \text{ is even and prime}\}$
   j) $\{n \in \mathbb{N} : n^2 - n + 1 > 0\}$

2. Given the sets $A = \{a,b,c\}$, $B = \{1,2,3\}$, and $C = \{A,B,C,D\}$, show

   a) $A \approx B$
   b) A and C are not equivalent.

3. Show that the union of two countable sets is countable.

4. For the following intervals, find an explicit one-to-one correspondence showing the intervals are equivalent.

   a) $\{a,b,c\} \approx \{1,2,3\}$
   b) $\{n \in \mathbb{N} : (n \le 50) \wedge (5 \mid n)\} \approx \{1,2,3,4,5,6,7,8,9,10\}$
   c) $[0,1] \approx [3,5]$
   d) $[0,1) \approx [0,\infty)$
   e) $(0,1) \approx \mathbb{R}$
   f) $[0,1] \approx [3,5]$

5. **(Even and Odd Natural Numbers)** Let E be the set of even positive integers and O be the set of odd positive integers. Given an explicit function to show the following equivalences.

a) $E \approx O$
b) $\mathbb{N} \approx O$
c) $N \approx E$
d) $E \approx \mathbb{Q}$
e) $O \approx \mathbb{Q}$

6. **(Power Sets)** If A, B are two sets, then $A \approx B \Rightarrow P(A) \approx P(B)$.

7. **($\approx$ as an Equivalent Relation)** Show that the relation "$\approx$" satisfies the following three conditions, called reflexive, symmetric, and transitive. If a relation satisfies these conditions, it is called an **equivalence relation** (We will study this type of relation in Chapter 3)

(i) $A \approx B$ (reflexive)
(ii) $A \approx B \Rightarrow B \approx A$ (symmetric)
(iii) $(A \approx B) \wedge (B \approx C) \Rightarrow A \approx C$ (transitive)

8. **(Cartesian Products)** Show $|A \times B| = |B \times A|$

9. **(Cardinality of Subsets)** Show for any two sets A, B, if $A \subseteq B$ then $|A| \leq |B|$.

10. **(Bijection from $\mathbb{N}$ to $\mathbb{Z}$)** Show the function $f : \mathbb{N} \rightarrow \mathbb{Z}$ is a bijection. Hint: One must show f is both one-to-one and onto. To show one-to-one, break the problem into two cases: n even and n odd and in either case let $f(m) = f(n)$ and show $m = n$.