Section 2.1 Basic Operations of Sets

**Purpose of Section** We present an informal discussion of sets, such as membership of sets, subsets, empty set, power set, union and intersection, and other fundamental properties. The material presented here will provide the background for further study.

**Introduction**

Sets are the most basic of all mathematical objects, simply collections of objects called *elements or members* of the set. Other synonyms are common for the word “set,” such as *collection, class, family, ensemble.* Hence, we can refer to a collection of people, family of fish, an ensemble of voters, and so on. We shall even consider sets whose members themselves are sets, such as the set of all classes at a university, or the family of all open intervals \((a,b)\) on the real line.

If a set does not contain too many members, one can specify the set by simply writing down the members inside a pair of brackets, such as

\[
\{\text{Terrance Tao, Ingrid Daubechies}\}
\]

or by

\[
\{2, 3, 5, 7, 13, 17, 19, 31, 61, 89\}
\]

which contains the first ten Mersenne primes. Sometimes sets contain an infinite number of elements, like the natural numbers where we might specify them by \(\{1,2,3,\ldots\}\), where the three dots after the 3 signify “and so on” and denotes the fact that the sequence of numbers is never ending.

One can also specify a set by specifying defining properties of the member of the set, such as

\[
\{x \in A : P(x)\}
\]

for all \(x \in A\) such that \(P(x)\) holds.
which reads “the set of all \( x \) in a set \( A \) such that condition \( P(x) \) is true.”

The set of even integers could be denoted by \( \{ x \in \mathbb{N} : x \text{ is an even integer} \} \). In the case when it is clear we are talking about natural numbers, we might simply write \( \{ x : x \text{ is an even integer} \} \). Some common sets in mathematics are the following.

<table>
<thead>
<tr>
<th>Common Sets in Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{N} = {1, 2, 3, \ldots } ) (the natural numbers or positive integers)</td>
</tr>
<tr>
<td>( \mathbb{Z} = { \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots } ) (integers)</td>
</tr>
<tr>
<td>( \mathbb{Q} = { x : x = p / q, p \text{ and } q \neq 0 \text{ are integers} } ) (rational numbers)</td>
</tr>
<tr>
<td>( \mathbb{R} ) = the set of real numbers</td>
</tr>
<tr>
<td>( \mathbb{C} ) = the set of complex numbers</td>
</tr>
</tbody>
</table>

\((a, b) = \{ x \in \mathbb{R} : a < x < b \} \) (open interval from \( a \) to \( b \))
\([a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \) (closed interval from \( a \) to \( b \))
\((a, \infty) = \{ x \in \mathbb{R} : x > a \} \) and \((\infty, a) = \{ x \in \mathbb{R} : x < a \} \) (open rays)
\([a, \infty) = \{ x \in \mathbb{R} : x \geq a \} \) and \((\infty, a] = \{ x \in \mathbb{R} : x \leq a \} \) (closed rays)

The half-open intervals \((a, b] \) and \([a, b)\) are defined similarly.

**Universe, Membership: Subsets, Equality, Empty Set**

- **Universe:** The universe \( U \) is the set consisting of the *totality of elements* under consideration in the current discussion. A common universe in number theory would be the natural numbers \( \mathbb{N} \), whereas in calculus common universes would be the set of real numbers \( \mathbb{R} \) or maybe some interval on the real line, like \([0, 1]\) or \([0, \infty)\).

- **Membership:** If an element \( x \) belongs to a set \( A \), we denote this by writing \( x \in A \), and if an element does not belong to \( A \) we write \( x \notin A \). For example, if \( A \) is the set of the top 10 mathematicians of all time, we would write Georg Cantor \( \in A \), Jerry Farlow \( \notin A \).

- **Subsets:** Often one is interested in a set \( A \) which is part of a larger set \( B \). We say that a set \( A \) is a *subset* of a set \( B \) if every element of \( A \) is also an element of \( B \). Symbolically, we write this as \( A \subseteq B \) and is read “\( A \) is contained
in $B$. If $A \subseteq B$ and $A \neq B$ we say that $A$ is a **proper subset** of a set $B$ and sometimes denote this as $A \subset B$.

Sets are often illustrated visually by **Venn diagrams**, where sets are represented as circles and elements of the set are points inside the circle. Figure 1 shows a Venn diagram which illustrates $A \subseteq B$.

![Venn Diagram Illustrating $A \subseteq B$](image)

**Figure 1**

**Margin Note**: $\begin{array}{cccc} N \subset Z \subset Q \subset R \subset C \end{array}$

**Equality of Sets**: Two sets $A$ and $B$ are **equal** if they consist of exactly the same elements. In other words

$$A = B \iff (x \in A \Rightarrow x \in B) \text{ and } (x \in B \Rightarrow x \in A)$$

Stated another way: $A = B \iff A \subseteq B$ and $B \subseteq A$.

**Complement of a Set**: If a set $A$ belongs to a universe $U$, the **complement** of $A$, written $\sim A$ consists of all members of $U$ that are *not* in $A$. That is

$$\sim A = \{x \in U : x \notin A\}$$

**Empty Set**: When most people think of sets they generally think of sets containing several elements, at least two generally. But in mathematics we often study sets that contain a single element, and sometimes none! The set with no elements is called the **empty set** (or **null set**) and denoted by the Greek letter $\emptyset$ (or sometimes the empty bracket $\{\}$). So why our interest in the empty set? Well, sets are generally determined by a specific property, and it is not always clear if there *are* elements that satisfy the property. For

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1 If you object to the importance of a set with no elements, you are like the persons who objected to the number 0, since it stands for nothing. The number 0 was resisted for centuries as a legitimate number.
example, consider the set consisting of the real numbers that satisfy the equation

\[ x^6 + 6x^4 + 11x^2 + 6 = 0 \]

If you don’t know algebra, there doesn’t seem anything wrong with this statement; however, if you do know algebra and rewrite the equation as

\[ (x^2 + 1)(x^2 + 2)(x^2 + 3) = 0 \]

you realize that the set is empty².

Note that we keep saying “the” empty set rather than “an” empty set. The reason for this language is because two sets are equal if they contain the same elements, and since the empty set contains no elements, there is only one empty set. Just remember, the empty set is not nothing, it is something, it is just that it contains nothing. You might think of the empty set as a bag that has nothing in it. In this regard, it is best to denote the empty set by \( \{ \} \) rather than \( \emptyset \). With this interpretation, it is clear that

\[ \emptyset = \{ \} \text{ is empty.} \]
\[ \{ \emptyset \} \text{ is not empty, it contains something, the empty set.} \]
\[ \{ \{ \emptyset \} \} \text{ is not empty, it contains the set that contains the empty set.} \]
\[ \{ \{ \{ \emptyset \} \} \} \text{ is not empty, it contains the set that contains the set that contains the empty set.} \]

And so on.

For example

\[ \{ \text{all people over 500 years old} \} = \{ x \in \mathbb{R} : x^2 + 1 = 0 \} \]

\( \in \) versus \( \subseteq \): When you ask if \( a \in A \) you ask the question is "a" a member of \( A \); when you ask if \( B \subseteq A \) you ask is every member of \( B \) a member of \( A \). For example \( \{ a, \{ a \} \} \not\in \{ a, \{ a \} \} \) but \( \{ a, \{ a \} \} \subseteq \{ a, \{ a \} \} \).

² Many problems in mathematics ask the existence of certain numbers or classes of numbers. The most famous unsolved problem in mathematics, Fermat’s Last Theorem, asked the question whether the set of integers \( \{ x, y, z \} \) satisfying \( x^n + y^n = z^n \) for integers \( n > 2 \) was empty. The set is empty, which was proven in 1993 by the English mathematician, Andrew Wiles.
Power Set: For every set $A$, the collection of all subsets of $A$ is called the power set of $A$ and denoted by $P(A)$. Suppose for example we have a small universe $U$ consisting of three elements $U = \{a, b, c\}$. We can readily see that there are 8 subsets of $U$ which occur in complementary pairs as follows.

<table>
<thead>
<tr>
<th>Set</th>
<th>Complement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>${a, b, c}$</td>
</tr>
<tr>
<td>${a}$</td>
<td>${b, c}$</td>
</tr>
<tr>
<td>${b}$</td>
<td>${a, c}$</td>
</tr>
<tr>
<td>${c}$</td>
<td>${a, b}$</td>
</tr>
</tbody>
</table>

Eight subsets of $\{a, b, c\}$
Table 1

**Rigor in Mathematics** In the study of mathematics, there should always be a sufficient degree of rigor, which means that equations and mathematical concepts should be logically precise. However, it is more important to understand basic principles and built up an intuition about the ideas. The great Swiss mathematician Leonard Euler had an uncanny intuition about concepts and although he often did not verify his thoughts, not one serious mathematician would ever say he wasn’t one of the greatest mathematicians who ever lived.

A few power sets of some other sets are given in Figure 2.

<table>
<thead>
<tr>
<th>Set</th>
<th>Power Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>${\emptyset}$</td>
</tr>
<tr>
<td>${a}$</td>
<td>${\emptyset, {a}}$</td>
</tr>
<tr>
<td>${a, b}$</td>
<td>${\emptyset, {a}, {b}, {a, b}}$</td>
</tr>
<tr>
<td>${a, {b}}$</td>
<td>${\emptyset, {a}, {{b}}, {a, {b}}}$</td>
</tr>
</tbody>
</table>

Power Sets
Table 2

**Theorem 1 (Guaranteed Subset)** For any set $A$, we have $\emptyset \subseteq A$.

**Proof** Since the goal is to show $x \in \emptyset \Rightarrow x \in A$ our job is done before we begin inasmuch as the hypothesis $x \in \emptyset$ of the implication is false since $\emptyset$ contains no elements. Hence, the proposition is true regardless of the set $A$. In other words $\emptyset$ is a subset of any set. (Which doesn’t mean it is a member of any set.)
Theorem 2  (Transitive Subsets)  Let $A$, $B$ and $C$ be sets. We have

$$(A \subseteq B) \land (B \subseteq C) \Rightarrow A \subseteq C.$$  

Proof: We will prove the conclusion $A \subseteq C$ and use the hypothesis as needed. Letting $x \in A$ the goal is to show $x \in C$. Since $x \in A$ and using the assumption $A \subseteq B$, we know $x \in B$. But the hypothesis also says $B \subseteq C$ and so we know $x \in C$. Hence, we have proved $A \subseteq C$, which proves the theorem. We illustrate this theorem with a Venn diagram shown in Figure 2.

Example 2  Power Set $P(A)$

Below are listed typical power sets.

<table>
<thead>
<tr>
<th>Set</th>
<th>Power set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = \emptyset$</td>
<td>$P(A) = {\emptyset}$</td>
</tr>
<tr>
<td>$A = {a}$</td>
<td>$P(A) = {\emptyset, {a}}$</td>
</tr>
<tr>
<td>$A = {a, b}$</td>
<td>$P(A) = {\emptyset, {a}, {b}, {a, b}}$</td>
</tr>
<tr>
<td>$A = {a, b, c}$</td>
<td>$P(A) = {\emptyset, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}$</td>
</tr>
</tbody>
</table>

Later, we will prove that for a set of $n$ elements the power set contains $2^n$ elements, whose proof looks like a candidate for induction$^3$.

Margin Note: Sometimes the power set of a set $A$ is denoted by $2^A$ since if a set contains $a$ elements, the power set has $2^a$ elements.

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$^3$ Things get a lot more interesting when we consider the family of all subsets of an infinite set like the natural numbers. It turns out that … well, we don’t want to ruin the fun for you.
Margin Note: One reason the concept of a set is so powerful is the fact that the elements can be anything, even sets themselves. In analysis they normally consist of sets of numbers, like the integers, real number, intervals on the real line, and so on. In geometry they are geometric objects, in probability they are sample spaces and events, and so on. In topology, one studies certain families of subsets of a given set, called the open sets of the set.

Example 3

Do you understand why all these are correct?

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$\emptyset \subseteq {x \in \mathbb{R} : x^2 = -1}$</td>
</tr>
<tr>
<td>b)</td>
<td>$3 \in \mathbb{N}$</td>
</tr>
<tr>
<td>c)</td>
<td>$-1 \notin \mathbb{N}$</td>
</tr>
<tr>
<td>d)</td>
<td>$\pi \notin \mathbb{Q}$</td>
</tr>
<tr>
<td>e)</td>
<td>$e \in \mathbb{R}$</td>
</tr>
<tr>
<td>f)</td>
<td>$3.5 \in \mathbb{C}$</td>
</tr>
<tr>
<td>g)</td>
<td>$3+2i \notin \mathbb{R}$</td>
</tr>
<tr>
<td>h)</td>
<td>${x : x^2 -1 = 0} \not\subset {x : x^3 -1 = 0}$</td>
</tr>
<tr>
<td>i)</td>
<td>${1,3,7,-1,\pi} \subseteq {-1,\pi,7,1,3}$</td>
</tr>
<tr>
<td>j)</td>
<td>${1,3,7} \subset {-1,\pi,7,1,3}$</td>
</tr>
</tbody>
</table>

If $A \subseteq B$ then every member of $A$ is also a member of $B$, but $A \in B$ means $A$ is a member of the set $B$, regardless of whether $A$ is a set of a single element. The following example tests your understanding of subsets and membership of sets.

Example 4 (Difference Between Subsets and Membership)

Remember the difference between membership in a set and a subset of a set. An element $x$ is a member of a set if it belongs to the set. A set is a subset if everything in the set belongs to the set. Those are two different questions.

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$\emptyset \in {\emptyset,{\emptyset}}$</td>
<td>Yes</td>
</tr>
<tr>
<td>b)</td>
<td>$\emptyset \subseteq {\emptyset,{\emptyset}}$</td>
<td>Yes</td>
</tr>
</tbody>
</table>
c) $\emptyset \in \{\{\emptyset\}\}$  
   Answer: No  

d) $a \in \{\{a\}, \{a, \{a\}\}\}$  
   Answer: No  

e) $\{a\} \in \{\{a\}, \{a, \{b\}\}\}$  
   Answer: Yes  

f) $\emptyset \subseteq \{\emptyset, \{\emptyset, \emptyset\}\}$  
   Answer: No  

g) $\{a, \{b\}\} \in \{a, \{a, \{b\}\}\}$  
   Answer: Yes  

h) $\{a, \{b\}\} \in \{a, \{a, b\}\}$  
   Answer: Yes  

i) $\{a, \{b\}\} \in \{\{b\}, a\}$  
   Answer: No  

j) $\{a, \{b\}\} \subseteq \{\{b\}, a\}$  
   Answer: Yes  

- a) is true since $\emptyset$ is a member of the set on the right.  
- b) is true since the empty set is a subset of any set, the reason being the empty set does not have any elements.  
- c) is false since $\emptyset$ is not a member of the set on the right.  
- d) is false since $a$ is not one of the elements of the set on the right.  

Margin Note: The empty set is a subset of any set.  

Theorem 3  (Power Set)  For $A$ and $B$ be sets, we have  

$$A \subseteq B \iff P(A) \subseteq P(B).$$  

Proof:  

$(A \subseteq B) \Rightarrow (P(A) \subseteq P(B))$  
We start by taking an arbitrary set $X \in P(A)$.  
Since $X \in P(A)$ we know $X \subseteq A$ and hence by hypothesis $A \subseteq B$ we gave $X \subseteq B$.  
But this means $X \in P(B)$ and so we conclude $P(A) \subseteq P(B)$.  

$(P(A) \subseteq P(B)) \Rightarrow (A \subseteq B)$  
We let $x \in A$.  
If $x \in A$, then $\{x\} \in P(A)$, and since by assumption $P(A) \subseteq P(B)$ we have $\{x\} \in P(B)$.  
But this means $x \in B$ and so we have proven $A \subseteq B$.  

Union, Intersection, and Difference of Sets
In traditional arithmetic and algebra, we carry out the binary operations of $+$ and $\times$ on numbers. In logic, we have the analogous binary operations of $\lor$ and $\land$ on sentences. In set theory we have the binary operations of union $\cup$ and intersection $\cap$ of sets, which in a sense are analogous\(^4\) to the ones in arithmetic and sentential logic.

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### Definition

- **Union:** The union of two sets $A$ and $B$, denoted $A \cup B$, is the set of elements that belong to $A$ or $B$ or both\(^5\). Symbolically

  $$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$

![Union Diagram]

- **Intersection:** The intersection of two sets $A$ and $B$, denoted $A \cap B$, is the set of elements that belong to $A$ and $B$. Symbolically

  $$A \cap B = \{ x : x \in A \text{ and } x \in B \}$$

![Intersection Diagram]

If $A \cap B = \emptyset$ then the sets $A$ and $B$ are called disjoint.

- **Difference:** The difference of two sets, denoted $A - B$, is defined to be the set of elements that belong to $A$ but not $B$. Symbolically

  $$A - B = \{ x \in U : x \in A \text{ and } x \notin B \}$$

---

\(^4\) If we were to define a Boolean algebra we would see that the binary operations of $\land, \lor$ are very analogous to those of $\cap, \cup$.

\(^5\) This “or” is the inclusive “or” in contrast to the exclusive “or” which means one or the other but not both.
Venn Diagrams

The concepts of union, intersection and relative complement of sets can be illustrated visually by use of Venn diagrams. Each Venn diagram begins with an oval representing the universal set, a set that contains all elements of in discussion, maybe the real number, complex numbers, and so on. Then, each set in the discussion is represented by a circle, where elements belonging to more than one set are placed in sections where circles overlap. Figure 2 illustrates typical Venn diagrams for two overlapping sets.

![Venn Diagrams for Two Sets](image)

Figure 2

Figure 3 illustrates Venn diagrams for three sets.

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Venn diagrams were the invention of British logician John Venn (1834-1923) who made major contributions to logic and probability. John Venn was an ordained minister but gave up the ministry in 1883 to concentrate on mathematics and logic.
Venn Diagrams for Three Sets
Figure 3

The properties of \( \cup \) and \( \cap \) in set theory have their counterparts in the properties of \( \lor \) and \( \land \) in sentential logic. Table 1 illustrates these counterparts.

<table>
<thead>
<tr>
<th>Tautology</th>
<th>Set Equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \lor Q \iff Q \lor P )</td>
<td>( A \cup B = B \cup A )</td>
</tr>
<tr>
<td>( P \land Q \iff Q \land P )</td>
<td>( A \cap B = B \cap A )</td>
</tr>
<tr>
<td>( P \lor (Q \lor R) \iff (P \lor Q) \lor R )</td>
<td>( A \cup (B \cup C') = (A \cup B) \cup C )</td>
</tr>
<tr>
<td>( P \land (Q \land R) \iff (P \land Q) \land R )</td>
<td>( A \cap (B \cap C') = (A \cap B) \cap C )</td>
</tr>
<tr>
<td>( P \lor (Q \lor R) \iff (P \lor Q) \land (P \lor R) )</td>
<td>( A \cup (B \cap C') = (A \cup B) \cap (A \cup C') )</td>
</tr>
<tr>
<td>( P \land (Q \land R) \iff (P \land Q) \lor (P \land R) )</td>
<td>( A \cap (B \cup C') = (A \cap B) \cup (A \cap C') )</td>
</tr>
<tr>
<td>( P \land P \iff P )</td>
<td>( A \cap A = A )</td>
</tr>
<tr>
<td>( P \lor P \iff P )</td>
<td>( A \lor A = A )</td>
</tr>
<tr>
<td>( P \land \neg (Q \land \neg Q) \iff P )</td>
<td>( A \land \emptyset = A )</td>
</tr>
<tr>
<td>( P \lor (Q \land \neg Q) \iff Q )</td>
<td>( A \cup \emptyset = A )</td>
</tr>
</tbody>
</table>

Equivalence between some Laws of Logic and Laws of Sets
Table 3

Naive versus Axiomatic Set Theory

Naive set theory, as we study in this section, studies basic properties of sets, such as complements, union, intersection, De Morgan’s laws, and so on using general intuition. Unfortunately, unless care is taken on exactly what
kind of collections of objects can be “accepted” as a set, it is possible to arrive at contradictions (i.e. Russell’s paradox), which we will learn about later in this chapter. Axiomatic set theory was created to place set theory on a firm axiomatic foundation where the axioms are consistent (no internal contradictions) and independent (no one axiom could be proven from the others). The most accepted axioms of set theory are the Zermelo–Fraenkel (ZF) axioms, named after logicians Ernst Zermelo (1871–1953) and Abraham Fraenkel (1891–1965).

Margin Note: The symbols ∪ and ∩ for set union and intersection is due to the Italian mathematician Giuseppe Peano (1858–1932).

**Theorem 4** For A and B sets, we have $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$.

**Proof:**
To show $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$ we show $\overline{B} \subseteq \overline{A}$ using the hypothesis $A \subseteq B$ as assistance when needed. Letting $x \in \overline{B}$ says that $x$ does not belong to $B$, but the hypothesis $A \subseteq B$ tells us that $A$ is contained in $B$ and hence $x$ does not belong to $A$ as well, or in other words $x \in \overline{A}$. Hence, we have proven $\overline{B} \subseteq \overline{A}$.

**Theorem 5:** Let $A, B$ and $C$ be sets. Then "∩" distributes over "∪". That is

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

**Proof:**
We show

a) ($\subseteq$): $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

b) ($\supseteq$): $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$

To show a) we write

$$x \in A \cap (B \cup C) \Rightarrow x \in A \text{ and } x \in B \cup C$$

$$\Rightarrow (x \in A) \text{ and } (x \in B \text{ or } x \in C)$$

But this means $x \in A \cap B$ or $x \in A \cap C$. In other words $x \in (A \cap B) \cup (A \cap C)$ which proves a).

To prove b) we argue a little differently. Since

$$B \subseteq B \cup C$$
$$C \subseteq B \cup C$$
we intersect each side with $A$, getting

$$A \cap B \subseteq A \cap (B \cup C)$$

$$A \cap C \subseteq A \cap (B \cup C)$$

and so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Thus b) is verified and so we have proven the desired result. □

**De Morgan’s Laws**

We saw in sentential logic the tautologies $(P \land Q) \equiv \neg P \lor \neg Q$ and

$$(P \lor Q) \equiv \neg P \land \neg Q.$$ The analogous identities for sets are called *De Morgan’s laws*.

**Theorem 6** Let $A$ and $B$ be sets. Prove De Morgan’s laws, which state

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

**Proof**

We prove the first De Morgan Law. We show

$$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$$

$(\subseteq)$ We let $x \in \overline{A \cup B}$.

$$x \in \overline{A \cup B} \Rightarrow x \notin A \cup B$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \in \overline{A} \text{ and } x \in \overline{B}$$

$$\Rightarrow x \in \overline{A} \cap \overline{B}$$

Hence $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

$(\supseteq)$ To show $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$, we let $x \in \overline{A \cap B}$.

$$x \in \overline{A \cap B} \Rightarrow x \in \overline{A} \text{ and } x \in \overline{B}$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \notin A \cup B$$

$$\Rightarrow x \in \overline{A \cup B}$$
Hence $A \cup B \supseteq \overline{A} \cap \overline{B}$. Hence, the first De Morgan Law is verified. The second De Morgan’s law is left to the reader. See Problem 9.
Problems

1. **(Set Notation)** Write the following sets in set notation \( \{x : P(x)\} \).
   
   a) The real numbers between 0 and 1.
   b) The natural numbers between 2 and 5.
   c) The set of prime numbers.
   d) \{1, 2, 3, ...\}
   e) \{5, 6, 7\}
   f) The solutions of the equation \( x^2 - 1 = 0 \).

2. **(Checking Subsets: True or False?)**
   
   a) \( \mathbb{Z} \subseteq \mathbb{R} \)
   b) \( \mathbb{R} \subseteq \mathbb{C} \)
   c) \((0, 1) \subseteq [0, 1]\)
   d) \((0, 1) \subseteq \mathbb{R}\)
   e) \((2, 5) \subseteq \mathbb{Q}\)
   f) \(\mathbb{Q} \subseteq (2, 5)\)
   g) \([1, 3] \subseteq \{1, 3\}\)
   h) \(\{1, 3\} \subseteq [1, 3]\)
   i) \(\{3, 15\} \subseteq \{3, 5, 7, 15\}\)

3. **(The Empty Set: True or False?)**
   
   a) \(\emptyset = \{\emptyset\}\)
   b) \(\emptyset \in \{\emptyset\}\)
   c) \(\emptyset \subseteq \{\emptyset\}\)
   d) \(A \cup \emptyset = A\)
   e) \(\emptyset = \{\emptyset\}\)
   f) \(\{\emptyset\} \in \{\{\emptyset\}\}\)
   g) \(\{\emptyset\} \in \emptyset, \{\emptyset\}\)

4. **True or False?**
   
   a) \(A \in A\)
   b) If \(A \subseteq B\) and \(x \notin B\) then \(x \notin A\)
c) If \( A \subseteq B \) then \( A \in B \).
d) If \( A \in B \) then \( A \subseteq B \).
e) If \( A \subseteq B \) and \( B \subseteq C \) then \( A \subseteq C \).
f) If \( A \subseteq B \) and \( B \subseteq C \) then \( A \subseteq C \).

5. **(Power Sets)** Find the power set of the given sets.

a) \( A = \{4, 5, 6\} \)
b) \( A = \{\oplus, \ominus, \otimes\} \)
c) \( A = \{a, \{b\}\} \)
d) \( A = \{a, \{b, \{c\}\}\} \)
e) \( A = \{a, \{a\}\} \)
f) \( A = \{\emptyset, \emptyset\} \)

6. **(Matching Sets)** Which pairs of the following sets are connected by one or more of the relations \( =, \in, \subseteq, \text{ or } \supseteq \)?

a) \( \mathbb{R} \)
b) \( 3 \)
c) \( \{1, 2, 3, ..., 10\} \)
d) \( \{x : x \text{ is an even integer}\} \)
e) \( (-1, 1) \)
f) \( \emptyset \)

7. **(Find the Set)** Let \( A = \{a_1, a_2, a_3, \ldots\} \) were \( a_n \) is the remainder of \( n \) divided by 5. List the elements of \( A \).

8. **(Interesting)** If \( A = \{a, b, c\} \) is \( A \in P(A) \)? Is \( A \subseteq P(A) \)?

9. **(Power Set as a Collection of Functions)** The power set of a set can be interpreted as the set of all functions\(^7\) defined on the set whose values are 0 and 1. For example, the functions defined on \( A = \{a, b\} \) with values 0 and 1 are

\[ f(a) = 0, f(b) = 0 \text{ corresponds to } \emptyset \]
\[ f(a) = 0, f(b) = 1 \text{ corresponds to } \{b\} \]
\[ f(a) = 1, f(b) = 0 \text{ corresponds to } \{a\} \]

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\(^7\) Although we haven’t introduced functions yet in this book, we are sure most readers have familiarity with the subject.
• \( f(a) = 1, f(b) = 1 \) corresponds to \( \{a, b\} \)

Show that the elements of the power set of \( A = \{a, b, c\} \) can be placed in this one-to-one correspondence with the functions on \( A \) whose values are either 0 or 1.

10. **(Second Power Set)** The second power set of a set \( A \) is the set of subsets of the set of subsets of the set, or \( P(P(A)) \). What is the second power set of \( A = \{a, b\} \)?

11. **(Axiom of Sets)** In the Zermelo–Fraenkel axioms of set theory, the Axiom of Power Sets states

\[
(\forall A)(\exists P(A))(\forall B)[B \in P(A) \iff (\forall C)(C \in B \Rightarrow C \in A)]
\]

where \( A, B, C \) are sets. State this axiom on English.

12. **(The Man Who Constructed Something from Nothing)** The German mathematician Leopold Kronecker once said, “God created the integers and all else is the work of man.” But logicians like to say that they created the integers. The German logician Gottlob Frege defined the integers from nothing; i.e. using only the empty set! How did he do it? He defined recursively:

\[
\begin{align*}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{1\} = \{\{\emptyset\}\} \\
3 &= \{2\} = \{\{\{\emptyset\}\}\} \\
& \quad \vdots
\end{align*}
\]

and so on. Show that in Kronecker’s system \( 2 \in 3 \) but \( 2 \notin 4 \). Any thoughts on how one would construct an “arithmetic” using this definition. How would you define “1 + 1” so you would get 2? What about “1 + 3”?

13. Let \( A, B \) and \( C \) be arbitrary subsets of a universe \( U \). Prove the following.

a) \( A \subseteq A \)
b) \( A \cup \emptyset = A \)
c) \( A \cap \emptyset = \emptyset \)
d) \( \emptyset = \overline{U} \)
Section 2.1  Basic Notions of Sets

14. (Find the Sets) Let the natural numbers \( \mathbb{N} \) be the universal set and 
\( E = \{2, 4, 6, \ldots\}, \; O = \{1, 3, 5, \ldots\}, \; F = \{5, 10, 15, \ldots\}, \; P = \{2, 3, 5, 7, 11, \ldots\} \) (prime numbers). Find the following.

a) \( E \cap O \)
b) \( E \cup F \)
c) \( \overline{E} \cup F \)
d) \( (P \cap \overline{F}) \cup E \)
e) \( (E \cup O) \cap P \)
f) \( (O \cap \overline{E}) \cup P \)

15. (Symmetric Difference) The symmetric difference to two sets \( A \) and \( B \) is the set of elements that belong to one of the sets but not both and is denoted by

\[
A \Delta B = (A - B) \cup (B - A).
\]

See Figure 4.

Show the following. Draw Venn diagrams to illustrate your proof.

a) \( A \Delta B = B \Delta A \)
b) \( A \Delta (B \Delta C) = (A \Delta B) \Delta C \)
c) \( A \cup B = (A \Delta B) \Delta (A \cup B) \)
d) \( A \cap B = \emptyset \iff A \Delta B = A \cup B \)
16. **(Difference Problems)** Simplify the following.

   a)   \[ A - (B - C) \]
   
   b)   \[ A - (B - (C \cup D)) \]

   c)   \[ A - (B - (C - D)) \]

   d)   \[ \overline{A} \cap (B \cup C) \]

17. **(NASC for Disjoint Sets)** Prove that a necessary and sufficient condition for \( A \) and \( B \) to be disjoint is \( A - B = A \).

18. **Distributive Law** Prove that if \( A, B \) and \( C \) are sets, then "\( \cup \)" distributes over "\( \cap \)". That is

   \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \].

19. **(Venn Diagram)** Prove \( A \subseteq B \iff A \cap \overline{B} = \emptyset \) and illustrate the proof with a Venn diagram.

20. **(Venn Diagram)** Prove \( A = B \iff \overline{A} = \overline{B} \) and illustrate the proof with a Venn diagram.

21. **(De Morgan’s Law)** Prove De Morgan’s Law

   \[ \overline{A \cap B} = \overline{A} \cup \overline{B} \].

22. **(Sets and Arithmetic)** Compare the set operations of union, intersection, and subtraction with the arithmetic operations of addition, multiplication and subtractions of numbers. In what ways are they similar? In what ways are they different?

23. **(Sets and Sentential Logic)** Compare the set operations of \( \cap, \cup, - \) with the sentential logic operations of \( \land, \lor, \lnot \). In what ways are they similar? In what ways are they different?

24. **(Computer Representation of Sets)** Finite sets can be represented efficiently by vectors of 0s and 1. For example suppose we are interested in subsets of a finite universe \( U = \{0, 1, 2, 3, 4, 5, 6, 7\} \). We represent a subset like \( A = \{2, 3, 6\} \) as a vector of 0s and 1’s with 1’s in the 2nd, 3rd, and 6th positions, and zeros elsewhere. That is 0110010. In the following problems, if a set \( A \subseteq U \) is provided, find the binary vector corresponding to the set; if a binary vector is given, find the corresponding subset of \( U \).
a) 1001111
b) 0000000
c) 1111111
d) \{1,5,7\}
e) \mathbb{U}
f) 1000111 + 1100000 (assume 1+ 1 = 1)
g) 0000001 + 1110000

25. **(Mystery Set)** A set \(A\) consists of integers divisible by 5 and a set \(B\) consists of those integers divisible by 3, and a set \(C\) consists of integers that are a multiple of 4. What is the set?