Section 1.4 Mathematical Proofs

Purpose of Section: Most theorems in mathematics take the form of an implication \( P \Rightarrow Q \) or as a biconditional \( P \Leftrightarrow Q \), where the biconditional can be verified by proving both \( P \Rightarrow Q \) and \( P \Leftarrow Q \). We will study a variety of ways of proving \( P \Rightarrow Q \) including a direct proof, proof by contrapositive, and three variations of proof by contradiction, including proof by reductio ad absurdum.

Introduction A theorem\(^1\) is a proposition, deduced from given assumptions, and the chain of reasoning used to deduce the proposition is called a proof of the theorem. Theorems are ultimately based on a collection of principles considered so self-evident that their truth value is taken as fact. Such accepted maxims are called axioms, and every area of mathematics, be it real or complex analysis, algebra, geometry, and even arithmetic is based on a collection of self-evident truths.

Types of Proofs
Most theorems in mathematics have the form of an implication \( P \Rightarrow Q \), where one assumes the validity of a sentence \( P \), then with the aid of existing mathematical facts, and aided by the laws of logic and reasoning, one arrives at the given conclusion \( Q \). Although the goal is always to “go from \( P \) to \( Q \),” there is more than one valid way of achieving this goal. Five equivalent ways, a direct proof, contrapositive, and three proofs by contradiction, one of which is called reduction ad absurdum, can be applied to prove the implication \( P \Rightarrow Q \).

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Five Ways to Prove \( P \Rightarrow Q \)
Table 1

The truth table in Table 2 verifies the equivalence of these five forms of the basic implication.

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\(^1\) The Hungarian mathematician Paul Erdos (1913–1996) once said that a mathematician is a machine for converting coffee into theorems.
In this and the next section we will demonstrate different methods of proof on a variety of theorems. Before presenting some theorems and proofs we begin by stating a few definitions. Definitions are very important in mathematics since they give precise meaning to concepts thus allowing people doing mathematics to communicate with the assurance their interpretations of concepts is the same as others.

**Basic Definitions:**

A **prime number** (or just prime) is a natural number 1, 2, 3, ⋯ greater than 1 that is divisible only by 1 and itself. Thus 1 is *not* a prime, 2 is, 3 is not, 4 is ⋯ The first 10 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. A number that is not prime is called a **composite** number. For example 4, 6, 8, 9, 10, ⋯ are composite.

An **odd integer**. An integer *n* is an odd integer (or just odd) that is not divisible by 2, or equivalently an integer *n* that can be written as *n* = 2\*k + 1 for some integer *k*. Thus, 1, 5, -7, 3, -311, are odd integers. Some odd natural numbers would be 1, 7, 9, 101, ⋯

An **even integer**. An integer *n* is an even integer (or just even) that is divisible by 2, or equivalently, an integer *n* that can be written as *n* = 2\*k for some integer *k*. Thus 0, 4, -8, 12, -96, ⋯ are even integers. Some even natural numbers would be 2, 10, 12, 26, ⋯

We begin with a very simple proof.

**Theorem 1: Direct Proof** If *n* is an odd natural number, then *n*² is odd.

**Proof:** Since *n* is odd it can be written in the form *n* = 2\*k + 1 for some integer *k*. Squaring both sides of this equation yields

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same truth values

Five Equivalent Sentences for Implication

Table 1
\[ n^2 = (2k + 1)^2 \]
\[ = 4k^2 + 4k + 1 \]
\[ = 2(2k^2 + 2k) + 1 \]

Since \( k \) is an integer so is \( s = 2k^2 + 2k \) and so we have \( n^2 = 2s + 1 \), which by definition shows \( n^2 \) is odd.

\[ \text{Theorem 2 (contrapositive)} \]

For any natural number \( n \) if \( 2^n - 1 \) is prime so is \( n \).

\[ \text{Proof} \]

By contrapositive we assume \( n = rs, 1 < s < r \) is a composite number (not prime) and observe
\[ x^r - 1 = x^{rs} - 1 \]
\[ = (x^r - 1)(x^{s(r-1)} + x^{s(r-2)} + \ldots + x^s + 1) \]

which means \( 2^n - 1 \) is composite since it is divisible by \( x^r - 1 \).

There are no magic bullets for proving theorems. Sometimes however the result to be proven provides the starting point, and the theorem can be proven by working backwards. The idea is to work backwards until one arrives at an obviously true statement, and then turn around and work to the end. The following theorem provides a good example of this technique.

\[ \text{Theorem 3 (Backwards Proof)} \]

Prove that for any two positive real numbers \( x \) and \( y \), one has
\[ \sqrt{xy} \leq \frac{x + y}{2} \]

\[ \text{Proof:} \]

We begin by writing the conclusion in equivalent algebraic form
\[ x + y \geq 2\sqrt{xy} \]

and using the fact that \( x, y \) are positive we have another equivalent form
\[ x + y - 2\sqrt{x\sqrt{y}} \geq 0 \]
or by factoring

\[(\sqrt{x} - \sqrt{y})^2 \geq 0.\]

But this statement is obviously true and equivalent to the result we wish to prove. Hence we have proven the theorem. □

**Margin Note:** A **conjecture** is a mathematical statement which is believed to be true but has not been proven. Once proven it is called a theorem. The **Goldbach conjecture** is one of the oldest unsolved problems in mathematics, which states that every even integer greater than 2 can be written as the sum of two (not necessarily distinct) primes. For example

4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 3 + 7 = 5 + 5, ..., and so on.

A famous conjecture that was proven in 2003 is the **Poincare conjecture** which was proven by the Russian mathematician Gregori Perelman. The proof is so technical that only a few specialists are able to appreciate it, and even then it took experts a few years of intensive study to verify its validity.

**Analysis of Proof Techniques**

- **Direct Proofs** \([P \implies Q]\)  A direct proof starts with the given assumption \(P\) and uses existing facts to establish the truth of the conclusion \(Q\).

**Indirect proofs**\(^2\) refer to proof by contrapositive or proof by contradiction which we introduce here

- **Contrapositive** \([\sim Q \implies \sim P]\)  In proof by contrapositive, one assumes the conclusion \(Q\) false, then proves the \(P\) false. One type of theorem where proof by contrapositive is often used is where the conclusion \(Q\) states something does not exist. Here it is convenient to assume the contrary; that the object in question does exist.

- **Contradiction** \([P \land \sim Q \implies \sim P], [(P \land \sim Q) \implies \sim Q]\)  In proof by contradiction, one assumes the hypotheses \(P\) to be true but the conclusion \(Q\) false, and from this reach some type of contradiction, either contradicting the assumption \(P\) or contradicting the denial \(\sim Q\). Proof by contradiction\(^3\) is a very common method of proof.

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\(^2\) The formal name for indirect proofs is *Modus tollens* (Latin for “mode that denies”).  
\(^3\) The English mathematician, G. H. Hardy, said proof by contradiction is one of a mathematician’s finest weapons.
• \textit{Reductio ad absurdum} \((P \land \neg Q) \Rightarrow (R \land \neg R)\) \textit{Reductio ad absurdum} is another form of proof by contradiction. After assuming \(P\) is true and \(Q\) is not true, one seeks to prove an absurd result (like \(1=0\) or \(x^2 < 0\)), which we denote by \(R \land \neg R\).

\begin{tcolorbox}
\textbf{Margin Note: Not Mathematical Proofs:}
\begin{itemize}
  \item The proof is so easy we’ll skip it.
  \item Don’t be stupid, of course it’s true!
  \item It’s true because I \textit{said} it’s true!
  \item scribble, scribble QED
  \item Oh God let it be true!
  \item I have this gut feeling.
  \item I did it last night.
  \item It works for 16.37.
  \item I \textit{define} it to be true!
\end{itemize}

You can think of a few yourself.
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In Theorem 2 we proved that the square of an odd natural number is odd. We now show the converse is also true: if \(n^2\) is odd so is \(n\).

\textbf{Theorem 4 (Proof by Contrapositive)}

\((\forall n \in \mathbb{N})(n^2 \text{ even } \Rightarrow n \text{ even})\)

\textbf{Proof:} Since it is easier to begin with a property of \(n\) and demonstrate the property holds for its square \(n^2\), this theorem is a good candidate for proof by contrapositive. Hence, we assume \(n\) odd and show \(n^2\) odd. But \(n\) odd means it can be written as \(n = 2k + 1\) for some natural number \(k\), and squaring gives \(n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1\). But \(s = 4k^2 + 4k\) is an integer and just \(n^2\) is odd. Thus, we have proven the contrapositive \(n \text{ odd } \Rightarrow n^2 \text{ odd}\), which is equivalent to the stated theorem \(n^2 \text{ even } \Rightarrow n \text{ even}\). \qed

\footnote{Hence the name reduction ad absurdum (reduction to the absurd).}
Who Has Proven the Stronger Theorem?

Jerry and Susan have each proven an important theorem with the same conclusion \( C \), and each hopes to win a Field’s Medal. But although the conclusions of their theorems are the same, their hypotheses are different. Jerry has assumed a hypothesis \( J \) so his theorem has the form \( J \Rightarrow C \), while Susan has assumed a hypothesis \( S \), so her theorem has the form \( S \Rightarrow C \). In their battle to see who has the stronger theorem (which theorem implies the other), Jerry makes the discovery that his hypothesis \( J \) is sufficient for Susan’s hypothesis \( S \). That is, \( J \Rightarrow S \) and hence he claims his theorem is the stronger theorem. Is Jerry correct? The answer is no! Susan’s weaker hypothesis means she has the stronger (or better) theorem as can be seen by the implication

\[
(J \Rightarrow S) \Rightarrow [(S \Rightarrow C) \Rightarrow (J \Rightarrow C)]
\]

The validity of this tautology is left to the reader. See Problem 7.

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Margin Note: There are theorems and then there are theorems. In the Classification Theorem for Simple Groups (known lovingly at the “enormous” theorem), the proof required the work of hundreds of mathematicians and consists of an aggregate of hundreds of papers. If the theorem were to be written out it is estimated it would take between 10,000 to 15,000 pages.

Look carefully at the following theorem \( P \Rightarrow Q \), whose proof is by contradiction \( (P \land \lnot Q) \Rightarrow Q \).

Theorem 5 \( \sqrt{2} \) is Irrational (Proof by Contradiction)

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5 The Fields Medal is regarded as the “Nobel Prize” in mathematics awarded every four years to one or more outstanding researchers under the age of forty. In the year 2006 there were four recipients of the award: they were Andrei Okounkov (Russia), Grigori Perelman (Russia), Terence Tao (Australia/USA), and Wendelin Werner (France). The author of this book can claim a relationship with one Field’s medalist, Charles Fefferman, who won the award in 1978. The author’s wife went to grade school with Professor Fefferman in the 1950s.
\[ \sqrt{2} \text{ is an irrational number}\]

**Proof** Assume the contrary, that is \( \sqrt{2} \) is rational, which means we can write \( \sqrt{2} = \frac{p}{q} \), where \( p \) and \( q \) are integers reduced to lowest form by canceling common factors in the numerator and denominator. Squaring both sides of this equation, gives

\[
\left(\sqrt{2}\right)^2 = \left(\frac{p}{q}\right)^2 \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2
\]

which means \( p^2 \) is even. But from Theorem 4 this implies \( p \) is even. But \( p \) even means it can be written \( p = 2k \) where \( k \) is an integer. Hence, the equation \( 2q^2 = p^2 \) can be written

\[
2q^2 = (2k)^2 = 4k^2
\]

which, dividing both sides by 2 yields \( q^2 = 2k^2 \) which implies \( q^2 \) is even and thus \( q \) is even. Hence, we have proven both \( p \) and \( q \) are even, thus contradicting the fact that we reduced the fraction \( p/q \) to lowest terms. Hence, we cannot assume \( \sqrt{2} = \frac{p}{q} \), thus proving that \( \sqrt{2} \) is irrational. □

### Historical Note:

The reader should know the story of Hippasus, the Pythagorean who first proved \( \sqrt{2} \) is irrational. The Pythagoreans were a religious sect that flourished in Samos, Greece around 500 B.C., founded by the Greek philosopher and mathematician Pythagoras. They believed the universe was based on whole numbers 1, 2, 3, ..., which stemmed from their observations from music and astronomy. When Hippasus proved \( \sqrt{2} \) was irrational, which according to legend was made at sea, the Pythagoreans considered this heresy and threw him overboard. So much for making one of the greatest mathematical discoveries of all time.

We now prove a famous theorem goes back to 300 B.C to the great Greek mathematician Euclid of Alexandria (present day Egypt) and relies on the fact that every natural number can be factored uniquely into the product of prime numbers. Although Euclid proved the theorem by *reductio ad absurdum*, the German mathematician Dirichlet later developed a direct proof using

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\(^6\) It may not be obvious but this theorem is of the form \( P \Rightarrow Q \). The theorem simply says \( \sqrt{2} \) is irrational so where is the “if” in the theorem? The “if” would define the meaning of the square root. We generally don’t say it here in the theorem since it is understood by everyone.
analytic function theory. Before proving the result we make one small observation: every natural number greater than 1 is divisible by a prime number. This is clear since every natural number $n$ can be written as the product of prime numbers and any factor in this product divides $n$. For example, the prime number 5 is divisible by a prime number, namely itself, while the composite number $30 = 2 \cdot 3 \cdot 5$ is divisible by three prime numbers, namely 2, 5, and 5.

**Note:** Indirect arguments or proof by contradiction are not foreign to our thinking. They go back to our earliest days. When a parent tells a child not to do something, the child makes an indirect argument when it says “I better not do it since if I did …”

**Theorem 6 (Infinite Number of Prime Numbers)**

There are an infinite number of prime numbers.

**Proof:** The proof is by contradiction. We assume there are only a finite number of prime numbers, which we enumerate $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_n$, where $p_n$ is the $n$th prime number. Now consider the product of these primes plus 1. That is

$$M = p_1 p_2 \cdots p_n + 1$$

Now, since $M$ is a natural number greater than 2, it must be divisible by a prime number, which we call $p$, which must be one of the given primes $p_1, p_2, \ldots, p_n$. But we claim $p$ is not any of these primes since none of $p_1, p_2, \ldots, p_n$ divide $M$ as easily be seen from the following equations were each quotient has a remainder of 1.

$$\begin{align*}
\frac{p_1 p_2 \cdots p_n + 1}{p_1} &= \frac{p_2 p_3 \cdots p_n + 1}{p_2} \\
\frac{p_1 p_2 \cdots p_n + 1}{p_3} &= \frac{p_1 p_2 p_4 \cdots p_n + 1}{p_4} \\
\vdots & \quad \vdots \\
\frac{p_1 p_2 \cdots p_n + 1}{p_n} &= \frac{p_1 p_2 \cdots p_{n-1} + 1}{p_{n-1}}
\end{align*}$$

Hence, $p$ is a prime number different from any of the primes $p_1, p_2, \ldots, p_n$ which contradicts the assumption these were the only primes. Hence, we must reject the assumption there are a finite number of primes and accept the fact there are an infinite number.
Note: One might be tempted to think \( M = 2 \cdot 3 \cdot 5 \cdots p_n + 1 \) is a prime for any \( n \), thus allowing a prime generating machine. For example

\[
2 \cdot 3 + 1 = 7 \text{ is prime, } 2 \cdot 3 \cdot 5 + 1 = 31 \text{ is prime, } 2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211 \text{ is prime}
\]

however \( 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509 \) is composite.

Margin Note: Many of the most important theorems in mathematics are proven by contradiction. There of the most famous are:

▶ Cantor’s seminal theorem that the real numbers are uncountable.
▶ Euclid’s proof that there are an infinite number of prime numbers.
▶ Pythagorean’s proof that \( \sqrt{2} \) is irrational.

Tips for Proving Theorems: Here are a few guidelines that might be useful for proving theorems.

▶ Draw figures to visualize the concepts.
▶ Construct examples that illustrate the general principles.
▶ Work backwards (i.e. what are the steps before the conclusion).
▶ For "\( \Rightarrow \)" theorems ask if the converse "\( \Leftarrow \)" is true.
▶ Modify the theorem to make it easier.
▶ Generalize i.e. does the theorem hold in more general situations?
  If a theorem is stated in two dimensions, does it hold in three dimension?
▶ Did you actually use all the assumptions?

Necessary and Sufficient Conditions

Many theorems in mathematics are of the form \( P \iff Q \), which means “\( P \) is true if and only if \( Q \) is true”, and stated as \( P \) is a necessary and sufficient condition for \( Q \). Since we have the logical equivalence

\[
P \iff Q \equiv (P \Rightarrow Q) \land (P \Leftarrow Q)
\]

theorems of this type are proven by proving both implications \( P \Rightarrow Q \) and \( Q \Rightarrow P \). The following theorem illustrates this idea.

Theorem 7 (If and Only If) Let \( n \) be any integer, then 3 divides \( n^2 - 1 \) if and only if 3 does not divide \( n \).

Proof We prove two implications.
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(3 divides \(n^2 - 1 \Rightarrow 3 \text{ does not divide } n\)): Since 3 is a prime number and divides \(n^2 - 1 = (n-1)(n+1)\) it must divide either \(n-1\) or \(n+1\). If 3 divides \(n-1\), it cannot divide \(n\) (it will have a remainder of 1), and if 3 divides \(n+1\) it cannot divide \(n\) (it will have a remainder of 2). Hence 3 does not divide \(n\).

(3 does not divide \(n \Rightarrow 3 \text{ divides } n^2 - 1\)): If 3 does not divide \(n\), then we can write

\[
\frac{n}{3} = q + \frac{r}{3}
\]

or \(n = 3q + r\), where the remainder \(r\) is either 1 or 2. If \(r = 1\) then \(n-1 = 3q\) which means 3 divides \((n-1)(n+1) = n^2 - 1\). If \(r = 2\) then \(n-2 = 3q\) or equivalently \(n-1 = 3r + 3 = 3(r+1)\), which also means 3 divides \(n^2 - 1\). □

**Lemmas and Corollaries:** In addition to theorems there are lemmas and corollaries. Although theorems, lemmas and corollaries are the same from a logical point of view, it is how they are used and their importance that distinguishes them.

- A lemma is a statement that is proven as an aid in proving a theorem. Often unimportant details are put in a lemma so not to clutter a theorem.
- A corollary is a statement that is easily deduced from a theorem.

Often an “if and only if” theorem can be proven without dividing it into two parts. The following example is such a theorem.

**Theorem 8 (if and only if)** If \(N\) is a natural number 1,2,3,..., then 3 divides \(N\) if and only if 3 divides the sum of the digits of \(N\).

**Proof:** We can represent \(N\) by

\[
N = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + 10a_1 + a_0
\]

and the sum by \(s = a_0 + a_1 + \cdots + a_k\). We now make the simple observation that the difference \(N - s\) between a number and the sum of its digits is divisible by 3, which can be seen by writing

\[
N - s = (a_k 10^k + a_{k-1} 10^{k-1} + \cdots + 10a_1 + a_0) - (a_k + a_{k-1} + \cdots + a_0)
\]

\[
= 999...9a_k + \cdots + 99a_2 + 9a_1
\]

\[
= 3(333...3a_k + \cdots + 3a_2 + 3a_1)
\]

\[
= 3m
\]

\[
\Rightarrow 3 \text{ divides } N
\]

\[
\Rightarrow 3 \text{ divides } s
\]

\[
\text{Therefore, } 3 \text{ divides } N \text{ if and only if } 3 \text{ divides } s.
\]
or simply \( \text{N} - s = 3m \), where \( m \) is a natural number.

(\( \Rightarrow \)) We now prove that if \( \text{N} \) is divisible by 3, so is the sum of its digits \( s \). If \( \text{N} \) is divisible by 3, we write \( \text{N} = 3j \), for \( j \) a natural number. Hence, we have \( 3j - s = 3m \), and solving for the sum \( s \) gives

\[
s = \text{N} - 3m = 3j - 3m = 3(j - m)\]

which proves that \( s \) is also divisible by 3.

(\( \Leftarrow \)) To prove the converse that if the sum of the digits \( s \) is divisible by 3, then so is the number \( \text{N} \), we begin by writing \( s = 3p \), where \( p \) is a natural number. Hence, our above equation \( \text{N} - s = 3m \) becomes \( \text{N} - 3p = 3m \), which if we solve for \( \text{N} \), gives

\[
\text{N} = 3p + 3m = 3(p + m)
\]

which shows \( \text{N} \) is divisible by 3.

Counterexample

At one time there was a theorem that stated \( 2^n + 1 \) was a prime number for natural numbers \( n \) until someone made the embarrassing observation that

\[
2^{32} + 1 = 4294967297 = 641 \times 6700417.
\]

Problems

1. **(Direct Proof)** Prove the following by a direct proof.

   a) The sum of two even integers is even.
   b) The sum of an even and odd integer is odd.
   c) If \( a \) divides \( b \), and \( b \) divides \( c \), then \( a \) divides \( c \).
   d) The product of two consecutive natural numbers plus the larger number is a perfect square. That is, the square of another natural number.
   e) Every odd integer is the difference between two perfect squares (the square of an integer).
   f) If \( a, b \) are real numbers, then \( a^2 + b^2 \geq 2ab \).
   g) The sum of two rational numbers is rational.
   h) Let \( p(x) \) be a polynomial and \( A \) is the sum of the coefficients of the even
powers, and \( B \) the sum of the coefficients of the odd powers. Show 
\[ A + B = p(1) \text{ and } A - B = p(-1). \]

2. **(Proof by Example)** Prove that \( \sqrt{10} \) is irrational.

3. **(Divisibility by 4)** Show that a natural number is divisible by 4 if and only if its last two digits are. For example 4 divides 256 and also divides 56.

4. **(Divisibility by 9)** Show that a natural number is divisible by 9 if and only if the sum of its digits is divisible by 9.

5. **(Proof by Contradiction)** Prove the following by contradiction
   a) If \( n \) is an integer and \( 5n + 2 \) is an even integer, then \( n \) is even.
   b) If \( x \) and \( y \) are integers and \( x + y \) is even, then \( x \) and \( y \) have the same parity (i.e. both are even or both are odd).
   c) Let \( x \) and \( y \) be integers. If \( xy \) is even, then at least one of \( x \) and \( y \) must be even.
   d) Prove that 1 is equal to 0.9999 \( \cdots \).

6. Prove the following theorems for integers \( m, n \).
   a) 5 divides \( n^4 - 1 \) if and only if 5 does not divide \( n \).
   b) 9 divides \( n \) if and only if 9 divides the sum of the digits of \( n \).
   c) The product \( mn \) is even if and only if at least one of \( m \) and \( n \) is even.

7. **(Clever Proof)** The rational number \( \frac{22}{7} \) is one of the most popular approximations to \( \pi \). Archimedes wrote the first known proof that \( \pi < \frac{22}{7} \), and many have been found since that time. One of the more ingenious proofs is the result of the inequality

\[
0 < \int_0^1 x^4 (1-x)^4 \, dx = \frac{22}{7} - \pi
\]

Evaluate the definite integral in this proof.

8. **(Counterexamples)** A counterexample is an exception to a rule. In mathematics, they are used to probe the boundaries of theorem. A
counterexample\(^7\) to a given claim may show that the assumptions are false or incomplete, and thus allow the mathematician to add or adjust the conjectures. Find counterexamples for the following faulty theorems and tell how you could add new hypothesis to make the claim valid.

a) If \( a > b \) then \( |a| > |b| \).

b) If \( (a - b)^2 = (m-n)^2 \) then \( a - b = m - n \).

c) If \( x \) and \( y \) are real numbers, then \( \sqrt{xy} = \sqrt{x}\sqrt{y} \).

d) If \( f \) is a continuous real-valued function defined on \([a,b]\), then there exists a \( c \in (a,b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

9. (Pythagorean Theorem) Prove the Pythagorean Theorem. That is, if \( a, b, c \) are the sides of a right triangle where \( c \) is the hypotenuse, then \( a^2 + b^2 = c^2 \). Use the diagram in Figure 1 and set the area of the larger square equal to the sum of the sub triangles and square inside it.

![Visual Proof of the Pythagorean Theorem](Image)

Figure 1

10. Comparing Theorems Verify the statement

\[
(J \Rightarrow S) \Rightarrow [(S \Rightarrow C) \Rightarrow (J \Rightarrow C)]
\]

which shows the theorem \( S \Rightarrow C \) (with the weaker hypothesis) is a stronger theorem than \( J \Rightarrow C \).

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11. **Comparing Theorems**  Verify that theorem $J \Rightarrow C_1$ is stronger (implies) than theorem $J \Rightarrow C_2$ if conclusion $C_1$ is stronger than conclusion $C_2$. In other words, show that

$$(C_1 \Rightarrow C_2) \Rightarrow [(J \Rightarrow C_1) \Rightarrow (J \Rightarrow C_2)]$$

is a tautology. This means if two theorems have the same hypothesis, then the theorem with the stronger conclusion is superior.

12. **Proof by Contradiction**  Verify the compound statement

$$Q \leftrightarrow \neg Q \Rightarrow [R \land \neg R]$$

by making a truth table.