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AN EXISTENCE THEOREM FOR PERIODIC SOLUTIONS OF A PARABOLIC BOUNDARY VALUE PROBLEM OF THE SECOND KIND*

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1. Introduction. It is proved that: if the coefficients of a second order parabolic equation in an infinite space time cylinder $D \times (-\infty, \infty)$, the nonhomogeneous term, and the mixed data on the boundary of $D \times (-\infty, \infty)$ are periodic in t with period T, then there exists a unique solution in $D \times (-\infty, \infty)$ which is also periodic in t with period T (the Dirichlet problem was solved by Shmulev [2]).

2. Results. The uniqueness and periodicity of the solutions can be proven a priori without using all the conditions needed for existence. The following theorems use the notation found in [1].

THEOREM 1. *Let $u = u(x, t)$ be a bounded solution of*

$$P: \quad \begin{cases} Lu = f(x, t), & (x, t) \in D \times (-\infty, \infty), \\ \dfrac{\partial u}{\partial \nu} + \beta(x, t)u(x, t) = g(x, t), & (x, t) \in \partial D \times (-\infty, \infty), \end{cases}$$

and assume that

(i) *L is uniformly parabolic in $\bar{D} \times (-\infty, \infty)$,*
(ii) *$c(x, t) \leqq 0$ for $(x, t) \in \bar{D} \times (-\infty, \infty)$,*
(iii) *$\beta(x, t) \leqq b_0 < 0$ for $(x, t) \in \partial D \times (-\infty, \infty)$,*
(iv) *∂D belongs to $C_{1+\lambda}$.*

If conditions (i)–(iv) *hold, then there exists at most one solution, $u = u(x, t)$, to the problem P.*

Proof. If we let u_1, u_2 be two bounded solutions of P, then using a theorem from [1, p. 147] one can show the a priori estimate

$$| e^t[u_1(x, t) - u_2(x, t)] | \leqq Ke^{t^*} \sup_{x \in \bar{D}} | u_1(x, t^*) - u_2(x, t^*) |$$

for all $(x, t) \in \bar{D} \times (-\infty, \infty)$ and t^* an arbitrary negative number. Since u_1, u_2 were assumed bounded, we can conclude that $u_1 = u_2$.

THEOREM 2. *If problem P possesses a unique solution $u = u(x, t)$ and if the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T, then the solution u is periodic in t with period T.*

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Proof. It is an easy matter to observe that $u(x, t + T)$ satisfies problem P and, hence, by Theorem 1, $u(x, t) \equiv u(x, t + T)$.

THEOREM 3 (Existence theorem for P). *The boundary value problem P has a unique bounded solution $u = u(x, t)$ which is periodic in t with period T provided:*

(i) *L is uniformly parabolic in $\bar{D} \times (-\infty, \infty)$;*
(ii) *the coefficients of L and f are uniformly Hölder continuous in the first variable and uniformly continuous in the second variable;*
(iii) *∂D belongs to class $C_{1+\lambda}$;*
(iv) *β and g are continuous on $\partial D \times (-\infty, \infty)$;*
(v) *$c(x, t) \leqq 0$ for $(x, t) \in \bar{D} \times (-\infty, \infty)$;*
(vi) *$\beta(x, t) \leqq b_0 < 0$ for $(x, t) \in \partial D \times (-\infty, \infty)$;*
(vii) *the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T.*

Proof. The proof uses the same technique used by Shmulev [2]. Consider a family of *initial* boundary value problems:

$$P_n : \begin{cases} Lu^n = f(x, t), & (x, t) \in D \times (t_n, t^*], \\ \dfrac{\partial u^n}{\partial \nu} + \beta(x, t)u^n(x, t) = g(x, t), & (x, t) \in \partial D \times (t_n, t^*], \\ u^n(x, t_n) = 0, & x \in \bar{D}, \end{cases}$$

where t^* is an arbitrary positive number. Nothing is known as to the convergence of $\{u^n\}$ on $\bar{D} \times (-\infty, \infty)$. However, given any $a < 0$, we can find a subsequence of $\{u^n\}$ which converges uniformly on $\bar{D} \times (a, \infty)$ to a function $u = u(x, t)$ which is a solution of P. To find this function $u = u(x, t)$ we first use the a priori estimate [1, p. 147] to obtain the following estimate on $\bar{D} \times [t_n, \infty)$:

$$| u^n(x, t) | \leqq K (\sup |f| + \sup |g|) \equiv C_0 < \infty,$$

the supremum of $|f|$ being taken over $\bar{D} \times (-\infty, \infty)$ and that of $|g|$ over $\partial D \times (-\infty, \infty)$. The constant K depends only on L, β and D. Next we pick integers $q > p > 0$ and observe that the difference $u^{p,q} \equiv u^q - u^p$ satisfies

$$\begin{aligned} &Lu^{p,q} = 0, && (x, t) \in D \times (t_p, \infty), \\ &\frac{\partial u^{p,q}}{\partial \nu} + \beta(x, t)u^{p,q}(x, t) = 0, && (x, t) \in \partial D \times (t_p, \infty), \\ &u^{p,q}(x, t_p) = u^q(x, t_p), && x \in \bar{D}. \end{aligned}$$

Making the transformation $v^{p,q} = e^t u^{p,q}$, we find that $v^{p,q}$ satisfies

$$Lv^{p,q} - v^{p,q} = 0, \qquad (x, t) \in D \times (t_p, \infty),$$

$$\frac{\partial v^{p,q}}{\partial \nu} + \beta(x, t)v^{p,q}(x, t) = 0, \qquad (x, t) \in \partial D \times (t_p, \infty),$$

$$\begin{aligned} v^{p,q}(x, t) &= e^{t_p}u^{p,q}(x, t_p) \\ &= e^{t_p}u^q(x, t_p), \qquad x \in \bar{D}. \end{aligned}$$

Again applying the a priori estimate [1], we estimate $v^{p,q}$ on $\bar{D} \times [t_p, \infty)$ by

$$| v^{p,q}(x, t) | \leqq Ke^{t_p} \sup_{x \in \bar{D}} | u^q(x, t_p) |.$$

But we have already seen that the elements of the sequence $\{u^n\}$ are bounded by C_0, so that $| v^{p,q}(x, t) | \leqq Ke^{t_p}C_0$ on $\bar{D} \times [t_p, \infty)$, where K and C_0 depend only on L, β, f, g and D. Substituting $u^{p,q}(x, t) = e^{-t}v^{p,q}(x, t)$, we have

$$| u^{p,q}(x, t) | \leqq KC_0e^{t_p - t}$$

for $(x, t) \in \bar{D} \times [t_p, \infty)$. From this it follows that on every subset $\bar{D} \times [a, \infty)$ the sequence $\{u^n\}$ is a uniform Cauchy sequence of continuous functions.

Defining $u(x, t)$ to be the pointwise limit of $\{u^n\}$, we now show that $u(x, t)$ satisfies P. To this end suppose that $v(x, t)$ is a function that satisfies

$$Lv = f(x, t), \qquad (x, t) \in D \times (t^*, \infty),$$

$$\frac{\partial v}{\partial \nu} + \beta(x, t)v(x, t) = g(x, t), \qquad (x, t) \in \partial D \times (t^*, \infty),$$

$$v(x, t^*) = u(x, t^*), \qquad x \in \bar{D},$$

where t^* is an arbitrary negative number. Choose n sufficiently large so that $t_n < t^*$. Letting $w^n = v - u^n$, we find that

$$Lw^n = 0, \qquad (x, t) \in D \times (t^*, \infty),$$

$$\frac{\partial w^n}{\partial \nu} + \beta(x, t)w^n(x, t) = 0, \qquad (x, t) \in \partial D \times (t^*, \infty),$$

$$w^n(x, t^*) = u(x, t^*) - u^n(x, t^*), \qquad x \in \bar{D}.$$

Applying the a priori estimate again, we have

$$| v(x, t) - u^n(x, t) | \leqq K \sup_{x \in \bar{D}} | u(x, t^*) - u^n(x, t^*) |$$

for $(x, t) \in \bar{D} \times [t^*, \infty)$ and $K = K(L, \beta, D)$. From this estimate, we see that the sequence $\{u^n\}$ converges to v uniformly on $\bar{D} \times [t,^* \infty)$, and since $\{u^n\}$ also converges to u in the same region, we conclude that $u = v$. That is, u satisfies

$$Lu = f(x, t), \qquad (x, t) \in D \times (t^*, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x, t)u(x, t) = g(x, t), \qquad (x, t) \in \partial D \times (t^*, \infty).$$

Since t^* was an arbitrary negative number, we conclude that u satisfies P. Finally the uniqueness and periodicity assertions follow from Theorems 1 and 2, respectively.

REFERENCES

[1] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[2] I. Shmulev, *Periodic solutions of the first boundary-value problem for parabolic equations*, Mat. Sb., 66 (1965), pp. 398–410.