A Sequence of Families Converging in an Equiconvergent Manner

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The theorem of Arzela-Ascoli states that a necessary and sufficient condition for a family \( B \) of real-valued continuous functions defined on a compact metric space \( X \) to be compact in \( C(X) \) is that the family \( B \) be uniformly bounded and equicontinuous. In showing that \( B \) is compact in \( C(X) \) the difficult part generally is to show that \( B \) is equicontinuous. The following theorem may be of some value in this regard. We first, however, state a definition.

**Definition:** If the pair \( (X,d) \) is a metric space, we say that a sequence of families of real-valued functions \( \{F_n\} = \{f_{\alpha}^n\} \), \( \alpha \) belonging to some index set \( A \), each \( \{f_{\alpha}^n\} \) defined on \( X \), converges to the family \( \{f_{\alpha}\} \) in an equiconvergent manner as \( n \to \infty \) if, for each \( \varepsilon > 0 \), there exists an integer \( N \), independent of \( \alpha \in A \), such that \( n \geq N \) implies

\[
|f_{\alpha}^n(x) - f_{\alpha}(x)| < \varepsilon
\]

for all \( x \in X \) and all \( \alpha \in A \).

**Theorem:** If for each \( n = 1, 2, ..., \) \( F_n = \{f_{\alpha}^n\} \) is a family of equicontinuous real-valued functions defined on a compact metric space \( (X,d) \) and if \( \{f_{\alpha}^n\} \) converges to \( \{f_{\alpha}\} \) in an equiconvergent manner as \( n \to \infty \), then the family \( \{f_{\alpha}\} \) is equicontinuous.

**Proof:** Since each family \( F_n \) is equicontinuous, we know that for every \( \varepsilon > 0 \) there exists a \( \delta_n > 0 \), independent of \( \alpha \), such that

\[
d(x,y) < \delta_n \Rightarrow |f_{\alpha}^n(x) - f_{\alpha}^n(y)| < \frac{\varepsilon}{3}
\]
for all $\alpha \in A$ from the definition. Also since $\{f^n_a\}$ converges to $\{f_a\}$ in an equiconvergent manner we know that if $\epsilon > 0$ then there exists an integer $N$ independent of $\alpha$ such that for all $x \in X, \alpha \in A$,

$$|f^n_a(x) - f_a(x)| < \frac{\epsilon}{3}$$

for all $n \geq N$. Now pick $\epsilon > 0$. We conclude that there exists a $\delta_n > 0$, independent of $\alpha \in A$, such that if $d(x, y) < \delta_n$, then for all $\alpha \in A$ we have

$$|f_a(x) - f_a(y)| \leq |f_a(x) - f^n_a(x)| + |f^n_a(x) - f^n_a(y)| + |f^n_a(y) - f_a(y)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

This completes the proof.