On q-Analogs of Multiple Zeta Values and other Multiple Harmonic Series

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Multiple Harmonic Sums

Example:

$$Z_n^{\geq}(s,t,u) := \sum_{i=1}^{n} \frac{1}{i^s} \sum_{j=1}^{i} \frac{1}{j^t} \sum_{k=1}^{j} \frac{1}{k^u} = \sum_{n\geq i\geq j\geq k\geq 1} i^{-s} j^{-t} k^{-u}$$

- $i \geq j \geq k$ are positive integers
- n may be finite or infinite ($0 \leq n \leq \infty$)
- s,t,u are positive integers ($s,t,u \in \mathbf{Z}^+$)
- $s > 1$ if $n = \infty$

With strict inequalities:

$$Z_n^{>}(s,t,u) := \sum_{i=1}^{n} \frac{1}{i^s} \sum_{j=1}^{i-1} \frac{1}{j^t} \sum_{k=1}^{j-1} \frac{1}{k^u} = \sum_{n\geq i> j> k\geq 1} i^{-s} j^{-t} k^{-u}$$

- $i > j > k$ are positive integers
- n may be finite or infinite ($0 \leq n \leq \infty$)
- s,t,u are positive integers ($s,t,u \in \mathbf{Z}^+$)
- $s > 1$ if $n = \infty$

$$Z_n^{\geq}(s_1,\dots,s_m) := \sum_{k_1=1}^{n} \frac{1}{k_1^{s_1}} \sum_{k_2=1}^{k_1} \frac{1}{k_2^{s_2}} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{1}{k_m^{s_m}} = \sum_{n\geq k_1\geq k_2\geq\cdots\geq k_m\geq 1} \prod_{j=1}^{m} k_j^{-s_j}$$

- $k_1 \geq k_2 \geq \cdots \geq k_m$ are positive integers
- n may be finite or infinite ($0 \leq n \leq \infty$)
- $s_1,\dots,s_m$ are positive integers ($\forall j, s_j \in \mathbf{Z}^+$)
- $s_1 > 1$ if $n = \infty$

$$Z_n^{>}(s_1,\dots,s_m) := \sum_{k_1=1}^{n} \frac{1}{k_1^{s_1}} \sum_{k_2=1}^{k_1-1} \frac{1}{k_2^{s_2}} \cdots \sum_{k_m=1}^{k_{m-1}-1} \frac{1}{k_m^{s_m}}$$

$$= \sum_{n \ge k_1 > k_2 > \cdots > k_m \ge 1} \prod_{j=1}^{m} k_j^{-s_j}$$

- $k_1 > k_2 > \cdots > k_m$ are positive integers
- n may be finite or infinite ($0 \le n \le \infty$)
- $s_1, \dots, s_m$ are positive integers ($\forall j, s_j \in \mathbf{Z}^+$)
- $s_1 > 1$ if $n = \infty$
- $\zeta(s_1,\dots,s_m) := Z_\infty^{>}(s_1,\dots,s_m)$ is called a *multiple zeta value* (MZV)

Relationship between $Z_n^{\ge}$ and $Z_n^{>}$

$$Z_n^{\ge}(s) = Z_n^{>}(s),$$

$$Z_n^{\ge}(s,t) = Z_n^{>}(s,t) + Z_n^{>}(s+t),$$

$$Z_n^{\ge}(s,t,u) = Z_n^{>}(s,t,u) + Z_n^{>}(s+t,u) + Z_n^{>}(s,t+u) + Z_n^{>}(s+t+u).$$

More generally, let $\vec{s} = (s_1, s_2, \dots, s_m)$. Then

$$Z_n^{\ge}(\vec{s}) = \sum Z_n^{>}(\vec{t}),$$

where the sum is over all $\vec{t}$ obtained from $\vec{s}$ by replacing any number of commas by plus signs.

Positive q-Integers

Definition 1 *The q-analog of $k \in \mathbf{Z}^+$ is*

$$[k]_q := \sum_{j=0}^{k-1} q^j$$

$$= 1 + q + q^2 + \cdots + q^{k-1}$$

$$= \begin{cases} \dfrac{1-q^k}{1-q}, & q \ne 1, \\ k, & q = 1. \end{cases}$$

Note that $\lim_{q \to 1} [k]_q = k$.

Can we find reasonable/interesting q-analogs of multiple harmonic sums?

q-Powers and q-Binomial Coefficients

Let $0 \le n \in \mathbf{Z}$ and $x, y \in \mathbf{R}$.

Define the asymmetric q-power by

$$(x+y)_q^n := \prod_{k=0}^{n-1} (x + yq^k).$$

Clearly,

$$\lim_{q \to 1} (x+y)_q^n = (x+y)^n.$$

The Gaussian binomial coefficient (a.k.a. the q-binomial coefficent) is defined for $0 \le n \in \mathbf{Z}$ and integer k by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \dfrac{(1-q)_q^n}{(1-q)_q^k \, (1-q)_q^{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

The q-Factorial

If we set

$$[n]!_q := \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q} = \frac{(1-q)_q^n}{(1-q)^n},$$

then evidently

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

and thus

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

We also have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{1-q^{n-k+j}}{1-q^j}, \quad 0 \le k \le n,$$

$$[n]!_q = \sum_{k=0}^{n(n-1)/2} a_k q^k,$$
$$a_k = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } k \text{ inversions}\}.$$

The q-Binomial Theorem

Theorem 2 (Finite q-Binomial Theorem) *Let $x, y \in \mathbf{R}$ and $0 \le n \in \mathbf{Z}$. Then*

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k.$$

Letting $q \to 1$, we deduce

Corollary 1 (Classical Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Corollary 2 (q-Vandermonde Convolution)

$$\begin{bmatrix} a+b \\ n \end{bmatrix} = \sum_{k=0}^{n} q^{(n-k)(a-k)} \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ n-k \end{bmatrix}.$$

Proof Sketch. Compare coefficients of y^n in

$$(1+y)_q^{a+b} = (1+y)_q^a \, (1+yq^a)_q^b.$$

□

Theorem 3 (G.-C. Rota) *Let q be a power of a prime positive integer. Then $\begin{bmatrix} n \\ k \end{bmatrix}$ is equal to the number of k-dimensional subspaces of the n-dimensional vector space*

$$\mathbf{F}_q^{\oplus n} = \underbrace{\mathbf{F}_q \oplus \cdots \oplus \mathbf{F}_q}_{n},$$

where $\mathbf{F}_q$ is the finite field with q elements.

Proof (Sketch). Fix $1 \le k \le n$. Any k-dimensional subspace is determined by a basis consisting of k linearly independent spanning vectors.

There are $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ bases for all k-dimensional subspaces.

Since different bases may span the same subspace, we divide by the number of possible choices of basis of a particular k-dimensional subspace:

$$(q^k-1)(q^k-q)\cdots(q^k-q^{k-1}).$$

The quotient is $\begin{bmatrix} n \\ k \end{bmatrix}$. □

The q-Exponential Function

If $0 < q < 1$, then

$$\lim_{n\to\infty} \begin{bmatrix} n \\ k \end{bmatrix} = \lim_{n\to\infty} \prod_{j=1}^{k} \frac{1-q^{n-k+j}}{1-q^j} = \prod_{j=1}^{k} \frac{1}{1-q^j}$$
$$= \frac{1}{(1-q)^k \, [k]!_q}.$$

Recall the q-binomial theorem in the form

$$(1+x)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad 0 \le n \in \mathbf{Z}.$$

Thus,

$$\prod_{j=0}^{\infty} (1+xq^j) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{[k]!_q} \left(\frac{x}{1-q}\right)^k.$$

Define the q-exponential function by

$$\exp_q(x) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} \, x^k}{[k]!_q} = (1+(1-q)x)_q^{\infty}.$$

Then

$$\lim_{q\to 1} \exp_q(x) = e^x, \quad D_q \exp_q(x) = \exp_q(qx).$$

The q-Gamma Function

The q-gamma function has infinite product representation

$$\Gamma_q(t) := \frac{(1-q)_q^\infty}{(1-q)^{t-1}(1-q^t)_q^\infty} = (1-q)^{1-t}\prod_{k=1}^{\infty}\frac{1-q^k}{1-q^{t+k-1}}.$$

For $t > 0$, we have also

$$\Gamma_q(t) := \int_0^\infty x^{t-1}\exp_q(-qx)\,d_qx.$$

It can be shown that $\lim_{q\to 1}\Gamma_q(t) = \Gamma(t)$.

Also $\Gamma_q(t+1) = [t]_q\Gamma_q(t)$, $\Gamma_q(1) = 1$, so

$$\Gamma_q(n+1) = [n]!_q, \quad 0 \le n \in \mathbf{Z}.$$

Askey proved a q-analog of the Bohr-Mollerup theorem: If $f(1) = 1$, $f(t+1) = [t]_q f(t)$ for some $0 < q < 1$, and $\log f$ is convex for $x > 0$, then $f(x) = \Gamma_q(x)$.

q-Harmonic Sums

Suppose we define

$$Z_n[s] := \sum_{n\ge k\ge 1}\frac{1}{[k]_q^s} = (1-q)^s\sum_{k=1}^{n}\frac{1}{(1-q^k)^s}.$$

If $0 < q < 1$, then $\lim_{k\to\infty}\frac{1}{(1-q^k)^s} = 1$.

Consequently, $Z_\infty[s] = \sum_{k=1}^{\infty}\frac{1}{[k]_q^s}$ diverges.

Options:

- Restrict n to be finite ($0 \le n < \infty$)
- Insist that $q > 1$ if $n = \infty$
- Insert convergence factors into the sum

eg. $\sum_{k=1}^{n}\frac{q^k}{[k]_q^s}$ or $\sum_{k=1}^{n}\frac{q^{sk}}{[k]_q^s}$ or $\sum_{k=1}^{n}\frac{q^{(s-1)k}}{[k]_q^s}$

L. Van Hamme:

$$\sum_{k=1}^{n}\frac{q^k}{1-q^k} = \sum_{k=1}^{n}\frac{(-1)^{k+1}q^{k(k+1)/2}}{1-q^k}\begin{bmatrix} n \\ k \end{bmatrix}.$$

K. Dilcher:

$$\sum_{k_1=1}^{n}\frac{q^{k_1}}{1-q^{k_1}}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{1-q^{k_2}}\cdots\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}} = \sum_{k=1}^{n}\frac{(-1)^{k+1}q^{k(k+1)/2+(m-1)k}}{(1-q^k)^m}\begin{bmatrix} n \\ k \end{bmatrix}.$$

Equivalently,

$$\sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}\prod_{j=1}^{m}\frac{q^{k_j}}{[k_j]_q} = \sum_{k=1}^{n}\frac{(-1)^{k+1}q^{k(k+1)/2+(m-1)k}}{[k]_q^m}\begin{bmatrix} n \\ k \end{bmatrix}.$$

Multiple q-Harmonic Sums

Definition 4 *Let n, m and $s_1, s_2, \ldots, s_m$ be positive integers. Define*

$$Z_n[s_1,\ldots,s_m] := \sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}\prod_{j=1}^{m}\frac{q^{k_j}}{[k_j]_q^{s_j}},$$

$$A_n[s_1,\ldots,s_m] := \sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}(-1)^{k_1+1} \times q^{k_1(k_1+1)/2}\begin{bmatrix} n \\ k_1 \end{bmatrix}\prod_{j=1}^{m}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Both sums vanish if $n = 0$. If $n > 0$ and $m = 0$, define $Z_n[\,] = A_n[\,] = 1$.

Abbreviations:

$\operatorname*{Cat}_{j=1}^{m}\{s_j\}$ denotes the sequence $s_1, s_2, \ldots, s_m$.

$$\{s\}^m := \operatorname*{Cat}_{j=1}^{m}\{s\} = \underbrace{s,\ldots,s}_{m}$$

(i.e. $m \ge 0$ consecutive copies of s).

Theorem 5 *Let* $n, r, a_1, b_1, \dots, a_r, b_r \in \mathbf{Z}^+$. *Then*

$$Z_n\Big[\operatorname*{Cat}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\Big] = A_n\Big[a_1, \{1\}^{b_1-1}, \operatorname*{Cat}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\Big].$$

Example 1 *Putting* $r=2$, $a_1=3$, $b_1=2$, $a_2=b_2=1$ *gives* $Z_n[1,1,3,1]=A_n[3,1,2]$, *i.e.*

$$\sum_{n\ge j\ge k\ge m\ge p\ge 1}\frac{q^{j+k+m+p}}{[j]_q[k]_q[m]_q^3[p]_q} = \sum_{n\ge k\ge m\ge p\ge 1}(-1)^{k+1}q^{k(k+1)/2}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{2k+p}}{[k]_q^3[m]_q[p]_q^2}.$$

Example 2 *Putting* $r=2$, $a_1=a_2=b_1=1$, $b_2=2$ *in Theorem 5 gives* $Z_n[2,2]=A_n[1,2,1]$, *i.e.*

$$\sum_{n\ge k\ge m\ge 1}\frac{q^{k+m}}{[k]_q^2[m]_q^2} = \sum_{n\ge k\ge m\ge p\ge 1}(-1)^{k+1}q^{k(k+1)/2}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^m}{[k]_q[m]_q^2[p]_q}.$$

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Recall Theorem 5:

$$Z_n\Big[\operatorname*{Cat}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\Big] = A_n\Big[a_1, \{1\}^{b_1-1}, \operatorname*{Cat}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\Big].$$

Corollary 3 *Let* $n, a, b \in \mathbf{Z}^+$. *Then*

$$Z_n\Big[\{1\}^{a-1}, b\Big] = A_n\Big[a, \{1\}^{b-1}\Big].$$

Proof. Put $r=1$ in Theorem 5. □

Example 3 *Putting* $b=1$ *and* $a=m$ *yields* $Z_n[\{1\}^m]=A_n[m]$, *which is Dilcher's result*

$$\sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}\prod_{j=1}^{m}\frac{q^{k_j}}{[k_j]_q} = \sum_{k=1}^{n}(-1)^{k+1}q^{k(k+1)/2}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{(m-1)k}}{[k]_q^m}.$$

Note the limiting case

$$\sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}\prod_{j=1}^{m}\frac{1}{k_j} = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^m}\binom{n}{k}.$$

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Recall Corollary 3: If $n, a, b \in \mathbf{Z}^+$ then

$$Z_n\Big[\{1\}^{a-1}, b\Big] = A_n\Big[a, \{1\}^{b-1}\Big].$$

Example 4 *Putting* $a=1$ *and* $b=m$ *yields* $Z_n[m]=A_n[\{1\}^m]$, *i.e.*

$$\sum_{k=1}^{n}\frac{q^k}{[k]_q^m} = \sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}(-1)^{k_1+1}q^{k_1(k_1+1)/2}\begin{bmatrix}n\\k_1\end{bmatrix}\prod_{j=1}^{m}\frac{1}{[k_j]_q},$$

with limiting cases

$$\sum_{k=1}^{n}\frac{1}{k^m} = \sum_{n\ge k_1\ge\cdots\ge k_m\ge 1}(-1)^{k_1+1}\binom{n}{k_1}\prod_{j=1}^{m}\frac{1}{k_j}$$

and

$$\sum_{k=1}^{\infty}\frac{q^k}{[k]_q^m} = \sum_{k_1\ge\cdots\ge k_m\ge 1}\frac{(-1)^{k_1+1}q^{k_1(k_1+1)/2}}{(1-q)_q^{k_1}}\prod_{j=1}^{m}\frac{1}{[k_j]_q}.$$

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Proof of Theorem 5

By induction, it suffices to establish the base cases

$$A_n[\,] = A_n[0] = 1 \text{ for } 0 < n \in \mathbf{Z}$$

and the following two recurrence relations:

Proposition 6 *Let* n, m *and* $s_1, s_2, \dots, s_m \in \mathbf{Z}^+$. *Then*

$$A_n[s_1,\dots,s_m] = \sum_{r=1}^{n}\frac{q^r}{[r]_q}A_r[s_1-1, s_2,\dots,s_m].$$

Proposition 7 *Let* n, m *and* $s_2, s_3, \dots, s_m \in \mathbf{Z}^+$. *Then*

$$A_n[0, s_2, s_3, \dots, s_m] = \frac{A_n[s_2-1, s_3,\dots,s_m]}{[n]_q}.$$

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Base Cases

The base case

$$A_n[\,] = 1 \text{ for } n > 0$$

is true by definition.

The other base case, namely

$$A_n[0] = 1 \text{ for } n > 0$$

is an easy consequence of the q-binomial theorem:

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k.$$

If $n > 0$, Putting $x = 1$ and $y = -1$ gives

$$\begin{aligned} A_n[0] &= \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \\ &= 1 - (1-1)_q^n \\ &= 1. \end{aligned}$$

Proof of Proposition 6 (1st recurrence for A_n)

We begin with the q-Pascal recurrence

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k \end{bmatrix} + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}.$$

Lemma 8 *Let k and n be positive integers with $k \le n$. Then*

$$\sum_{r=k}^{n} q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = q^k \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. Write the q-Pascal recurrence in the form

$$q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} r \\ k \end{bmatrix} - \begin{bmatrix} r-1 \\ k \end{bmatrix},$$

multiply through by q^k, and sum on r. □

Definition 9 *Let n, m, and $s_1, \ldots, s_m$ be non-negative integers. Define*

$$W_n[s_1, \ldots, s_m] := \sum_{n \ge k_1 \ge \cdots \ge k_m \ge 1} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

If $n = 0$, the sum is empty and $W_0[\vec{s}] = 0$. If $n > 0$ and $m = 0$, define $W_n[\,] := 1$.

We can now write

$$\begin{aligned} &A_n[s_1, \ldots, s_m] \\ &= \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{(s_1-1)k}}{[k]_q^{s_1}} W_k[s_2, \ldots, s_m] \\ &= \sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k-1)/2 + (s_1-1)k} W_k}{[k]_q^{s_1}} \boxed{q^k \begin{bmatrix} n \\ k \end{bmatrix}}. \end{aligned}$$

By Lemma 8, the boxed expression is $\displaystyle\sum_{r=k}^{n} q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}$.

It follows that

$$\begin{aligned} &A_n[s_1, \ldots, s_m] \\ &= \sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k-1)/2 + (s_1-1)k} W_k}{[k]_q^{s_1}} \sum_{r=k}^{n} q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \\ &= \sum_{r=1}^{n} q^r \sum_{k=1}^{r} \frac{(-1)^{k+1} q^{k(k-1)/2 + (s_1-1)k} W_k}{[k]_q^{s_1}} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \\ &= \sum_{r=1}^{n} \frac{q^r}{[r]_q} \sum_{k=1}^{r} \frac{(-1)^{k+1} q^{k(k+1)/2 + (s_1-2)k} W_k}{[k]_q^{s_1-1}} \begin{bmatrix} r \\ k \end{bmatrix} \\ &= \sum_{r=1}^{n} \frac{q^r}{[r]_q} A_r[s_1 - 1, s_2, \ldots, s_m]. \end{aligned}$$

□

In the penultimate step, we used

$$\frac{[r]_q}{[k]_q} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} r \\ k \end{bmatrix}, \quad 1 \le k \le r.$$

Multiple Zeta Values

$$\zeta(s_1,\ldots,s_m) := \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^{m} k_j^{-s_j}.$$

The multiple series is absolutely convergent if

$$\sum_{j=1}^{n} \Re(s_j) > n, \qquad n = 1,2,\ldots,m.$$

Euler ($m = 2$):

$$2\zeta(s,1) = s\zeta(s+1) - \sum_{j=1}^{s-2} \zeta(s-j)\zeta(j+1),$$

where $2 \le s \in \mathbf{Z}$.

Period One

For all non-negative integers n,

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\{4\}^n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!},$$

$$\zeta(\{6\}^n) = \frac{6\cdot(2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(\{8\}^n) = \frac{8\cdot(2\pi)^{8n}}{(8n+4)!}, \times\left\{\left(1+\frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1-\frac{1}{\sqrt{2}}\right)^{4n+2}\right\}.$$

More generally, let $k \in \mathbf{Z}^+$ and $\omega := e^{i\pi/k}$. Then

$$\sum_{n=0}^{\infty} (-1)^n x^{2kn} \zeta(\{2k\}^n) = \prod_{j=0}^{k-1} \frac{\sin(\pi x \omega^j)}{\pi x \omega^j}.$$

Period Two

For all non-negative integers n,

$$\zeta(\{3,1\}^n) = 4^{-n}\zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!},$$

$$\zeta(3,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n} \zeta(4k+3)\zeta(\{4\}^{n-k}) = \sum_{k=0}^{n} \frac{2\pi^{4k}}{(4k+2)!}\left(-\frac{1}{4}\right)^{n-k} \zeta(4n-4k+3),$$

$$\zeta(2,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n} (-1)^k \zeta(\{4\}^{n-k})\Big\{(4k+1) \times \zeta(4k+2) - 4\sum_{j=1}^{k} \zeta(4j-1)\zeta(4k-4j+3)\Big\}.$$

The study of multiple zeta values touches on

- Number theory: algebraic independence
- Knot theory: polynomials of Kaufman, Kontsevich
- Quantum field theory: Feynman diagrams
- Algebra: Quasi-symmetric functions, K-theory
- Combinatorics: shuffles, quad-trees
- Symbolic computation: Gröbner bases
- Numerical computation: hunting for or disproving the existence of identities

• Partial fractions: Euler, Nielsen, Y. Ohno

• Series transformations: Subbarao, Ramanujan

• Linear algebra: D. Bailey, R. Girgensohn

• Contour integration: P. Flajolet & B. Salvy

• Special functions: J. & D. Borwein, DB

• Differential equations: D. Bowman, DB

• Harmonic algebra: M. Hoffman

• Lyndon Bases: D. Broadhurst, M. Bigotte

• Analytic Continuation: T. Arakawa, M. Kaneko, S. Akiyama, S. Egami, Y. Tanigawa, J. Zhao

• Combinatorics: H. Minh, J. Ryoo, DB

• Tate Motives: T. Terasoma, A. Goncharov

• Knot Theory: D. Kreimer, T. Le, J. Murakami

Multiple q-Zeta Values

In analogy with

$$Z_n^{\geq}[s_1,\ldots,s_m] = \sum_{n\geq k_1\geq\cdots\geq k_m\geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}},$$

define

$$Z_n^{>}[s_1,\ldots,s_m] = \sum_{n\geq k_1>\cdots>k_m\geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

Then

$$Z_\infty^{>}[s_1,\ldots,s_m] = \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}$$

is a q-analog of the multiple zeta value

$$\zeta(s_1,\ldots,s_m) = \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^m \frac{1}{k_j^{s_j}}.$$

M. Kaneko investigated analytic properties of the Riemann q-zeta function

$$\zeta[s] := \sum_{k=1}^{\infty} \frac{q^{tk}}{[k]_q^s}, \qquad t = s-1.$$

This suggests we consider

$$\zeta[s_1,\ldots,s_m] := \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Recall we previously defined

$$Z_\infty^{>}[s_1,\ldots,s_m] = \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

Let $\vec{s} = (s_1,\ldots,s_m)$. Then

$$\zeta[\vec{s};q] = q^{|\vec{s}|} Z_\infty^{>}[\vec{s};1/q], \qquad |\vec{s}| := \sum_{j=1}^m s_j.$$

Stuffle Multiplication

Example.

$$\zeta(a)\zeta(b,c) = \sum_{n>0} n^{-a} \sum_{m>k>0} m^{-b}k^{-c}$$

$$\begin{aligned} &= \sum_{n>m>k>0} n^{-a}m^{-b}k^{-c} \\ &\quad + \sum_{n=m>k>0} n^{-a-b}k^{-c} \\ &\quad + \sum_{m>n>k>0} m^{-b}n^{-a}k^{-c} \\ &\quad + \sum_{m>n=k>0} m^{-b}k^{-a-c} \\ &\quad + \sum_{m>k>n>0} m^{-b}k^{-c}n^{-a} \end{aligned}$$

$$\begin{aligned} &= \zeta(a,b,c) + \zeta(a+b,c) + \zeta(b,a,c) \\ &\quad + \zeta(b,a+c) + \zeta(b,c,a). \end{aligned}$$

Let u and v be (ordered) lists of positive integers. Then

$$\zeta(u)\zeta(v) = \sum_{w \in u*v} \zeta(w),$$

where $u * v$ is the multiset defined by the recursion

$$\begin{aligned}(s,u)*(t,v) \\ &= (s, u*(t,v)) \cup (t,(s,u)*v) \cup (s+t, u*v),\end{aligned}$$

$$\varepsilon * u = u * \varepsilon = u,$$

where ε is the empty list.

eg.

$$\begin{aligned} a*(b,c) &= (a, \varepsilon*(b,c)) \cup (b, a*c) \cup (a+b, \varepsilon*c) \\ &= \{(a,(b,c)), (b,a,\varepsilon*c), (b,c,a*\varepsilon), \\ &\qquad (b,a+c,\varepsilon*\varepsilon), (a+b,c)\} \\ &= \{(a,b,c), (b,a,c,), (b,c,a), \\ &\qquad (b,a+c), (a+b,c)\},\end{aligned}$$

whence

$$\begin{aligned}\zeta(a)\zeta(b,c) &= \zeta(a,b,c) + \zeta(b,a,c) + \zeta(b,c,a) \\ &\quad + \zeta(b,a+c) + \zeta(a+b,c).\end{aligned}$$

Counting Stuffles

Let $f(|u|,|v|)$ denote the number of lists in $u * v$.

The recursive decomposition implies that the generating function

$$F(x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m,n) x^m y^n$$

satisfies the functional equation

$$F(x,y) = 1 + xF(x,y) + yF(x,y) + xyF(x,y).$$

It follows that

$$F(x,y) = (1-x-y-xy)^{-1},$$

$$\begin{aligned} f(m,n) &= \sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{\min(m,n)} \binom{n}{k}\binom{m}{k} 2^k \\ &= \left|\left\{(b_1,\dots,b_m) \in \mathbf{Z}^m : \sum_{j=1}^{m} |b_j| \le n\right\}\right|.\end{aligned}$$

Explicit Description

Abbreviate $\{1,2,\dots,n\}$ by $\langle n\rangle$.

Define a *stuffle* on $(m,n) \in \mathbf{Z}^+ \times \mathbf{Z}^+$ as a pair (ϕ,ψ) of order-preserving injective mappings

$$\phi : \langle m\rangle \to \langle m+n\rangle, \quad \psi : \langle n\rangle \to \langle m+n\rangle$$

such that the union of their images is equal to $\langle r\rangle$ for some r with $\max(m,n) \le r \le m+n$.

Stuffle multiplication can now be written in the form

$$\begin{aligned}\zeta(s_1,\dots,s_m)\zeta(t_1,\dots,t_n) \\ &= \sum_{(\phi,\psi)} \zeta\Big(\operatorname*{Cat}_{k=1}^{r} \left\{s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)}\right\}\Big),\end{aligned}$$

where the sum is over all stuffles (ϕ,ψ) on (m,n), and $r = r(\phi,\psi)$ is as above.

$$\begin{aligned}\frac{q^{(s+t-2)k}}{[k]_q^{s+t}} &= \frac{q^{(s+t-1)k}}{[k]_q^{s+t}}\left\{1 + \frac{1}{q^k} - 1\right\} \\ &= \frac{q^{(s+t-1)k}}{[k]_q^{s+t}} + \frac{q^{(s+t-1)k}}{[k]_q^{s+t-1}} \cdot \frac{(1-q^k)}{q^k} \\ &= \frac{q^{(s+t-1)k}}{[k]_q^{s+t}} + (1-q)\frac{q^{(s+t-2)k}}{[k]_q^{s+t-1}}.\end{aligned}$$

$$\begin{aligned}\zeta[s]\zeta[t] &= \sum_{k>0} \frac{q^{(s-1)k}}{[k]_q} \sum_{j>0} \frac{q^{(t-1)j}}{[j]_q} \\ &= \sum_{k>j>0} \frac{q^{(s-1)k+(t-1)j}}{[k]_q^s [j]_q^t} \\ &\quad + \sum_{j>k>0} \frac{q^{(t-1)j+(s-1)k}}{[j]_q^t [k]_q^s} + \sum_{k>0} \frac{q^{(s+t-2)k}}{[k]_q^{s+t}} \\ &= \zeta[s,t] + \zeta[t,s] + \zeta[s+t] \\ &\qquad + (1-q)\zeta[s+t-1].\end{aligned}$$

q-Stuffles

More generally, expanding the product

$$\zeta[s_1,\ldots,s_m]\zeta[t_1,\ldots,t_n]$$

yields sums of products of terms of the form

$$\frac{q^{(s-1)k+(t-1)j}}{[k]_q^s[j]_q^t},$$

which, if $k=j$, reduces to

$$\frac{q^{(s+t-2)k}}{[k]_q^{s+t}}=(1-q)\frac{q^{(s+t-2)k}}{[k]_q^{s+t-1}}+\frac{q^{(s+t-1)k}}{[k]_q^{s+t}}.$$

It follows that

$$\zeta[s_1,\ldots,s_m]\zeta[t_1,\ldots,t_n]$$
$$=\sum_{(\phi,\psi)}\sum_A(1-q)^{|A|}\zeta\Big[\operatorname*{Cat}_{k=1}^{r}\big\{s_{\phi^{-1}(k)}+t_{\psi^{-1}(k)}-(k\in A)\big\}\Big],$$

where the outer sum is over all stuffles (ϕ,ψ) on (m,n), the inner sum is over all subsets A of the intersection of the images of ϕ and ψ, $r=|\phi(\langle m\rangle)\cup\psi(\langle n\rangle)|$ as before, and the Boolean expression $(k\in A)$ takes the value 1 if true (i.e. $k\in A$) and 0 if false (i.e. $k\notin A$).

Period-1 Sums Reduce

Theorem 10 *If n is a positive integer and $s>1$, then*

$$n\zeta[\{s\}^n]$$
$$=\sum_{k=1}^{n}(-1)^{k+1}\zeta[\{s\}^{n-k}]\sum_{j=0}^{k-1}\binom{k-1}{j}(1-q)^j\zeta[ks-j].$$

Example 5 *With $n=2$, we get*

$$2\zeta[s,s]=\zeta[s]\zeta[s]-\big(\zeta[2s]+(1-q)\zeta[2s-1]\big).$$

Corollary 4 *If n is a positive integer and $s>1$, then*

$$n\zeta(\{s\}^n)=\sum_{k=1}^{n}(-1)^{k+1}\zeta(\{s\}^{n-k})\zeta(ks).$$

Let $\mathfrak{S}_n$ denote the group of $n!$ permutations of $\langle n\rangle=\{1,2,\ldots,n\}$.

Theorem 11 *Let n be a positive integer, and let $s_j>1$ for $1\le j\le n$. Then*

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\Big[\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\Big]=\sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{k=1}^{|\mathcal{P}|}(|P_k|-1)!$$
$$\times\sum_{\nu_k=0}^{|P_k|-1}\binom{|P_k|-1}{\nu_k}(1-q)^{\nu_k}\zeta[p_k-\nu_k],$$

where the outer sum on the right is over all unordered set partitions $\mathcal{P}=\{P_1,\ldots,P_m\}$ of $\langle n\rangle$, $1\le m=|\mathcal{P}|\le n$, and $p_k=\sum_{j\in P_k}s_j$.

Corollary 5 (M. Hoffman)

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\Big(\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\Big)$$
$$=\sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{P\in\mathcal{P}}(|P|-1)!\,\zeta\Big(\sum_{j\in P}s_j\Big).$$

Parity Reduction

Theorem 12 *Let $m\in\mathbf{Z}^+$ and let $s_1,\ldots,s_m$ be real numbers with $s_1>1$, $s_m>1$, and $s_j\ge 1$ for $1<j<m$. Then*

$$\zeta\Big[\operatorname*{Cat}_{k=1}^{m}s_k\Big]+(-1)^m\zeta\Big[\operatorname*{Cat}_{k=1}^{m}s_{m-k+1}\Big]$$

can be expressed as a $\mathbf{Z}[q]$-linear combination of multiple q-zeta values of depth less than m.

That is, the coefficients in the linear combination are polynomials in q with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

A Double Generating Function

Theorem 13

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^{m+1}v^{n+1}\zeta[m+2,\{1\}^n]$$
$$= 1 - \exp\left\{\sum_{k=2}^{\infty}\left\{u^k + v^k - \big(u+v+(1-q)uv\big)^k\right\} \times \frac{1}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\}.$$

Corollary 6 *If* $0 \le m, n \in \mathbf{Z}$, *then*

$$\zeta[m+2,\{1\}^n] = \zeta[n+2,\{1\}^m].$$

Corollary 7 (q-Euler) *Let* $0 \le m \in \mathbf{Z}$. *Then*

$$2\zeta[m+2,1] = (m+2)\zeta[m+3] + (1-q)\,m\,\zeta[m+2] - \sum_{k=2}^{m+1}\zeta[m+3-k]\,\zeta[k].$$

In particular, when $m=0$ we get $\zeta[2,1]=\zeta[3]$.

The proof of Theorem 13 makes essential use of the basic hypergeometric function

$${}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}\,\middle|\, x\right] = \sum_{n=0}^{\infty}\frac{(1-q^a)_q^n(1-q^b)_q^n}{(1-q^c)_q^n(1-q)_q^n}\,x^n, \quad |x|<1.$$

Routine series manipulations reveal that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta[m+2,\{1\}^n] = 1 - {}_2\phi_1\left[\begin{matrix} q^{-y}, q^x \\ q^{1+x} \end{matrix}\,\middle|\, q^{1+y}\right]$$

Heine's q-analog

$${}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}\,\middle|\, q^{c-a-b}\right] = \frac{\Gamma_q(c)\Gamma_q(c-a-b)}{\Gamma_q(c-a)\Gamma_q(c-b)}$$

of Gauss's ${}_2F_1$ summation formula then gives

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta[m+2,\{1\}^n] = 1 - \frac{\Gamma_q(1+x)\Gamma_q(1+y)}{\Gamma_q(1+x+y)}.$$

If $\vec{s} = (s_1,\ldots,s_m)$ define

$$\text{weight}(\vec{s}) := |\vec{s}| = \sum_{j=1}^{m} s_j,$$
$$\text{depth}(\vec{s}) := m,$$
$$\text{height}(\vec{s}) := \#\{j : s_j \ge 2\}.$$

Theorem 14 (J. Okuda & Y. Takeyama)

$$1 + (w-uv)\sum_{s,m,h\ge 0} u^{s-m-h}v^{m-h}w^{h-1}\sum_{\substack{\text{weight}(\vec{s})=s\\ \text{depth}(\vec{s})=m\\ \text{height}(\vec{s})=h}}\zeta[\vec{s}]$$
$$= \exp\left\{\sum_{k=2}^{\infty}\left(u^k+v^k-\alpha^k-\beta^k\right)\frac{1}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\},$$

where α *and* β *satisfy the equations*

$$\alpha+\beta = u+v+(q-1)(w-uv), \qquad \alpha\beta = w.$$

———————————-

Theorem 13 is case $w=0$ *of Theorem 14.*

The Simplex Integral

M. Kontsevich: If $s_1,\ldots,s_m \in \mathbf{Z}^+$, then

$$\zeta(s_1,\ldots,s_m) = \int\prod_{k=1}^{m}\left(\prod_{r=1}^{s_k-1}\frac{dt_r^{(k)}}{t_r^{(k)}}\right)\frac{dt_{s_k}^{(k)}}{1-t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0,$$

and is abbreviated (D. Broadhurst) by

$$\int_0^1 \prod_{k=1}^{m} A^{s_k-1}B, \qquad A = \frac{dt}{t}, \quad B = \frac{dt}{1-t}.$$

$$\zeta(2,1) = \sum_{n>m>0} n^{-2}m^{-1} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(k+j)^{-2}k^{-1}$$

$$= \sum_{k=1}^{\infty} k^{-1}\sum_{j=1}^{\infty}(k+j)^{-1}\int_0^1 t^{k+j-1}\,dt$$

$$= \sum_{k=1}^{\infty} k^{-1}\int_0^1 t^{-1}\sum_{j=1}^{\infty}\int_0^t u^{k+j-1}\,du\,dt$$

$$= \int_0^1 t^{-1}\int_0^t (1-u)^{-1}\sum_{k=1}^{\infty} k^{-1}u^k\,du\,dt$$

$$= \int_0^1 t^{-1}\int_0^t (1-u)^{-1}\sum_{k=1}^{\infty}\int_0^u v^{k-1}\,dv\,du\,dt$$

$$= \int_{1>t>u>v>0} \frac{dt}{t}\cdot\frac{du}{1-u}\cdot\frac{dv}{1-v}$$

$$= \int_0^1 AB^2.$$

The Jackson Integral

Suppose $f:(0,b]\to\mathbf{R}$ and $0<x\le b$.

The Jackson q-integral of f on the subinterval $(0,x]$ is

$$\int_0^x f(t)\,d_qt := (1-q)\sum_{j=0}^{\infty} xq^j f(xq^j).$$

If there exists $0\le\alpha<1$ such that $|f(t)t^{\alpha}|$ is bounded on $(0,b]$, then the integral converges to a function $F(x)$ on $(0,b]$.

Additionally, F is a q-antiderivative of f:

$$D_qF(x) := \frac{F(qx)-F(x)}{(q-1)x} = f(x), \quad 0<x\le b.$$

Note that

$$\lim_{q\to 1} D_qF(x) = F'(x),$$

and

$$\lim_{q\to 1}\int_0^x f(t)\,d_qt = \int_0^x f(t)\,dt.$$

The Jackson Simplex Integral

Let $s_1,\ldots,s_m$ are positive integers. Recall:

$$\zeta(s_1,\ldots,s_m) = \int\prod_{k=1}^{m}\left(\prod_{r=1}^{s_k-1}\frac{dt_r^{(k)}}{t_r^{(k)}}\right)\frac{dt_{s_k}^{(k)}}{1-t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1>t_1^{(1)}>\cdots>t_{s_1}^{(1)}>\cdots>t_1^{(m)}>\cdots>t_{s_m}^{(m)}>0.$$

Theorem 15

$$\zeta[s_1,\ldots,s_m] = \int\prod_{k=1}^{m}\left(\prod_{r=1}^{s_k-1}\frac{d_qt_r^{(k)}}{t_r^{(k)}}\right)\frac{d_qt_{s_k}^{(k)}}{y_k-t_{s_k}},$$

where

$$y_k := \prod_{j=1}^{k} q^{1-s_j},$$

and the integral is over the same simplex as above.

Duality

Let $a_i,b_i\in\mathbf{Z}^+$ and $k=\sum_{i=1}^{n}(a_i+b_i)$. Then

$$\zeta(a_1+1,\{1\}^{b_1-1},\ldots,a_n+1,\{1\}^{b_n-1})$$

$$= \int_0^1\prod_{i=1}^{n} A^{a_i}B^{b_i}$$

$$= \int_{1>t_1>\cdots>t_k>0}\prod_{j=1}^{k} f_j(t_j)\,dt_j$$

$$= \int_{1>u_k>\cdots>u_1>0}\prod_{j=1}^{k} f_j(u_j)\,du_j, \quad u_j = 1-t_j$$

$$= \int_0^1\prod_{i=n}^{1} A^{b_i}B^{a_i}$$

$$= \zeta(b_n+1,\{1\}^{a_n-1},\ldots,b_1+1,\{1\}^{a_1-1}).$$

Generalized Duality

Definition 16 *Let n and $s_1, \ldots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S(s_1, \ldots, s_n; m) := \sum_{\substack{c_1, \ldots, c_n \geq 0 \\ c_1 + \cdots + c_n = m}} \zeta(s_1 + c_1, \ldots, s_n + c_n).$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \left\{a_i + 1, \{1\}^{b_i - 1}\right\},$$
$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \left\{b_i + 1, \{1\}^{a_i - 1}\right\}.$$

Theorem 17 (Y. Ohno) *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have the equality*

$$S(\vec{s}; m) = S(\vec{s}\,'; m).$$

Generalized q-Duality

Definition 18 *Let n and $s_1, \ldots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S[s_1, \ldots, s_n; m] := \sum_{\substack{c_1, \ldots, c_n \geq 0 \\ c_1 + \cdots + c_n = m}} \zeta[s_1 + c_1, \ldots, s_n + c_n].$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \left\{a_i + 1, \{1\}^{b_i - 1}\right\}$$
$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \left\{b_i + 1, \{1\}^{a_i - 1}\right\}.$$

Theorem 19 *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have*

$$S[\vec{s}; m] = S[\vec{s}\,'; m].$$

q-Duality

Corollary 8 *If $\vec{s}$, $\vec{s}\,'$ are dual argument lists, then*

$$\zeta[\vec{s}] = \zeta[\vec{s}\,'].$$

In other words, if $a_i, b_i \in \mathbf{Z}^+$ $(1 \leq i \leq n)$, then

$$\zeta\Big[\operatorname*{Cat}_{i=1}^{n} \left\{a_i + 1, \{1\}^{b_i - 1}\right\}\Big] = \zeta\Big[\operatorname*{Cat}_{i=n}^{1} \left\{b_i + 1, \{1\}^{a_i - 1}\right\}\Big].$$

Proof. Put $m = 0$ in Theorem 19 (generalized q-duality). □

q-Sum Formula

Definition 20 *Let $t_1, \ldots, t_n$ be positive integers.*

$$\zeta^*[t_1, \ldots, t_n] := \zeta\Big[t_1 + 1, \operatorname*{Cat}_{j=2}^{n} t_j\Big].$$

Corollary 9 (q-Sum Formula) *For any integers $0 < k \leq n$, we have*

$$\sum_{t_1 + t_2 + \cdots + t_n = k} \zeta^*[t_1, t_2, \ldots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers $t_1, \ldots, t_n$ with sum equal to k.

Proof. If we take the dual argument lists in the form $\vec{s} = (n+1)$ and $\vec{s}\,' = (2, \{1\}^{n-1})$ and put $m = k - n$, then Theorem 19 states that

$$\zeta[k+1] = \sum_{\substack{c_1, \ldots, c_n \geq 0 \\ c_1 + \cdots + c_n = k-n}} \zeta\Big[2 + c_2, \operatorname*{Cat}_{j=2}^{n} \{1 + c_j\}\Big]$$
$$= \sum_{\substack{t_1, \ldots, t_n \geq 1 \\ t_1 + \cdots + t_n = k}} \zeta\Big[t_1 + 1, \operatorname*{Cat}_{j=2}^{n} t_j\Big].$$

□

q-Cyclic Sum Formula

Definition 21 *Let* $s_j \in \mathbf{Z}^+$ *for* $1 \le j \le n$ *and put* $\vec{s} = (s_1, \dots, s_n)$. *Let* σ *denote the* n*-cycle* $(1\,2 \cdots n)$, *and let*

$$\mathcal{C}(\vec{s}) := \{(s_{\sigma^j(1)}, \dots, s_{\sigma^j(n)}) : 1 \le j \le n\}$$

denote the set of cyclic permutations of $\vec{s}$.

Recall the definition

$$\zeta^*[s_1, \dots, s_n] := \zeta[s_1 + 1, s_2, \dots, s_n].$$

Theorem 22 *Let* $\vec{s}$ *and* $\vec{s}\,'$ *be dual argument lists. Then*

$$\sum_{\vec{t} \in \mathcal{C}(\vec{s})} \zeta^*[\vec{t}\,] = \sum_{\vec{t} \in \mathcal{C}(\vec{s}\,')} \zeta^*[\vec{t}\,].$$

Proof of Generalized q-Duality

Let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y.

Let $\mathfrak{h}^0$ denote the subalgebra $\mathbf{Q}1 \oplus x\mathfrak{h}y$.

The $\mathbf{Q}$-linear map $\widehat{\zeta}$ is defined on $\mathfrak{h}^0$ by

$$\widehat{\zeta}[1] := \zeta[\,] = 1$$

and

$$\widehat{\zeta}\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}\Big] = \zeta\Big[\operatorname*{Cat}_{i=1}^{s}\big\{a_i + 1, \{1\}^{b_i - 1}\big\}\Big],$$

for positive integers a_i, b_i $(1 \le i \le s)$.

Let τ be the anti-automorphism of $\mathfrak{h}$ that switches x and y.

Then q-duality simply says that

$$\widehat{\zeta}[\tau w] = \widehat{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.$$

For each $n \in \mathbf{Z}^+$, let D_n be the derivation on $\mathfrak{h}$ that maps $x \mapsto 0$ and $y \mapsto x^n y$.

Let θ be an indeterminate (formal parameter).

Define

$$\Delta := \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n \quad \text{and} \quad \sigma := \exp(\Delta).$$

Then

Δ is a derivation on $\mathfrak{h}[[\theta]]$, and
σ is an automorphism of $\mathfrak{h}[[\theta]]$.

For any word $w \in \mathfrak{h}^0$, define

$$f[w;\theta] := \widehat{\zeta}[\sigma w]$$

and

$$g[w;\theta] := f[\tau w;\theta] = \widehat{\zeta}[\sigma\tau w].$$

Recall

$$D_n \text{ sends } x \mapsto 0,\ y \mapsto x^n y.$$

Thus,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n : x \mapsto 0,\ y \mapsto \{\log(1 - x\theta)^{-1}\}y$$

and

$$\sigma = \exp(\Delta) : x \mapsto x,\ y \mapsto (1 - x\theta)^{-1}y.$$

Therefore,

$$f\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta\Big] = \widehat{\zeta}\Big[\prod_{i=1}^{s} x^{a_i}\big\{(1 - x\theta)^{-1}y\big\}^{b_i}\Big]$$

$$= \sum_{m=0}^{\infty} \theta^m \sum_{\substack{c_1,\dots,c_n \ge 0 \\ c_1 + \cdots + c_n = m}} \zeta\Big[\operatorname*{Cat}_{j=1}^{n}\{t_j + c_j\}\Big],$$

$$= \sum_{m=0}^{\infty} \theta^m S[\vec{t}; m],$$

where

$$\vec{t} = (t_1, \dots, t_n) = (\operatorname*{Cat}_{i=1}^{s}\{a_i + 1, \{1\}^{b_i - 1}\}), n = \sum_{i=1}^{s} b_i.$$

Generalized q-Duality Reformulated

Theorem 23 *For all* $w \in \mathfrak{h}^0$, $f[w;\theta] = g[w;\theta]$, *i.e.* $\widehat{\zeta}\circ\sigma$ *is invariant under ordinary duality* τ.

Theorem 24 *Let* $a_i, b_i \in \mathbf{Z}^+$ *and* $\sum_{i=1}^s (a_i+b_i) > 2$. *Let* $\theta' := q\theta - 1$, *and set*

$$I^m = \underbrace{\{0,1\}\times\cdots\times\{0,1\}}_{m}.$$

The generating functions f *and* g *satisfy the difference equation*

$$\sum_{\substack{\epsilon,\delta\in I^s\\ \delta_1<a_1,\epsilon_s<b_s}} (-\theta)^{\overline{\delta}\cdot\overline{\epsilon}}(1-q)^{\delta\cdot\epsilon} f\Big[\prod_{i=1}^s x^{a_i-\delta_i}y^{b_i-\epsilon_i};\theta\Big]$$
$$= \sum_{\substack{\delta,\epsilon\in I^{s+1}\\ \delta_{s+1}=\epsilon_1=0\\ \delta_1<a_1,\epsilon_{s+1}<b_s}} (-\theta')^{\overline{\delta}\cdot\overline{\epsilon}-1}(1-q)^{\delta\cdot\epsilon} f\Big[\prod_{i=1}^s x^{a_i-\delta_i}y^{b_i-\epsilon_{i+1}};\theta'\Big].$$

Here, $\overline{\delta}$ *to denotes the ordered tuple whose* i^{th} *component is* $1-\delta_i$, *and of course* $\delta\cdot\epsilon$ *denotes the dot product* $\sum_i \delta_i\epsilon_i$. *Similarly,* $\overline{\epsilon}$ *denotes the ordered tuple whose* i^{th} *component is* $1-\epsilon$, *and* $\overline{\delta}\cdot\overline{\epsilon} = \sum_i(1-\delta_i)(1-\epsilon_i)$.

Proof of Theorem 23. Use induction on the total degree of the word $\prod_{i=1}^s x^{a_i}y^{b_i}$.

The base case is clearly satisfied, since the word xy is self-dual.

Now apply Theorem 24 to f and g and subtract the resulting two equations.

The terms whose words have total degree less than $\sum_{i=1}^s(a_i+b_i)$ are cancelled by the induction hypothesis.

This leaves us with

$$(-\theta)^s\Big\{f\Big[\prod_{i=1}^s x^{a_i}y^{b_i};\theta\Big] - g\Big[\prod_{i=1}^s x^{a_i}y^{b_i};\theta\Big]\Big\}$$
$$= (-\theta')^s\Big\{f\Big[\prod_{i=1}^s x^{a_i}y^{b_i};\theta'\Big] - g\Big[\prod_{i=1}^s x^{a_i}y^{b_i};\theta'\Big]\Big\}.$$

Thus, the function

$$H(\theta) := (-\theta)^s\Big\{f\Big[\prod_{i=1}^s x^{a_i}y^{b_i};\theta\Big] - g\Big[\prod_{i=1}^s x^{a_i}y^{b_i};\theta\Big]\Big\}$$

satisfies the functional equation $H(\theta) = H(\theta')$, where $\theta' = q\theta - 1$.

One can show that $H(\theta)$ is a meromorphic function of θ of the form

$$H(\theta) = \theta^s \sum_{n=1}^\infty \frac{h_n}{[n]_q - \theta q^n},$$

with at worst simple poles at $\theta = p_n := q^{-n}[n]_q$ for positive integers n.

Note that

$$0 = p_0 < p_1 < p_2 < \cdots < p_{n-1} < p_n < \cdots$$

and

$$p_n{}' = qp_n - 1 = p_{n-1} \quad \text{for all } n \ge 1.$$

The functional equation

$$H(\theta) = H(\theta')$$

thus implies that if H has a pole at p_n, then H must also have a pole at p_{n-1}.

Since H has no pole at p_0, it follows that each $h_n = 0$.

Thus, H vanishes identically and so $f = g$. □

Derivations

Definition 25 (K. Ihara & M. Kaneko) *Define a derivation on* $\mathfrak{h}$ *for each positive integer* n *by*

$$\partial_n(x) = x(x+y)^{n-1}, \qquad \partial_n(y) = -x(x+y)^{n-1}y.$$

Theorem 26 (Ihara & Kaneko) *For all positive integers* n *and words* $w \in \mathfrak{h}^0$, $\widehat{\zeta}(\partial_n(w)) = 0$.

Theorem 27 (q-Analog) *For all positive integers* n *and words* $w \in \mathfrak{h}^0$, $\widehat{\zeta}[\partial_n(w)] = 0$.

Theorem 27 is actually *equivalent* to generalized q-duality (Theorem 23).

Proof of Theorem 27

Proof. Let $\sigma = \exp(\Delta)$, $\tilde\sigma = \tau\sigma\tau$,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n, \qquad \partial = \sum_{n=1}^{\infty} \frac{\partial_n}{n}\theta^n.$$

Generalized q-duality (Theorem 23): $\forall w \in \mathfrak{h}^0$,

$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w] = \widehat{\zeta}[\tau\sigma\tau w] \iff (\sigma - \tilde\sigma)w \in \ker\widehat{\zeta}$.

We show that in fact, $(\sigma - \tilde\sigma)\mathfrak{h}^0 = \partial\mathfrak{h}^0$.

To prove this, we require the following identity of Ihara and Kaneko.

Proposition 28 $\exp(\partial) = \tilde\sigma\sigma^{-1}$.

To complete the proof of Theorem 27, observe that since

$$\partial = \log\left(\tilde\sigma\sigma^{-1}\right) = \log\left(1 - (\sigma - \tilde\sigma)\sigma^{-1}\right)$$

$$= -(\sigma - \tilde\sigma)\sum_{n=1}^{\infty}\frac{1}{n}\Big((\sigma - \tilde\sigma)\sigma^{-1}\Big)^{n-1}\sigma^{-1},$$

and

$$\sigma - \tilde\sigma = \left(1 - \tilde\sigma\sigma^{-1}\right)\sigma = \Big(1 - \exp(\partial)\Big)\sigma$$
$$= -\partial\sum_{n=1}^{\infty}\frac{\partial^{n-1}}{n!}\sigma,$$

we see that

$$\partial\mathfrak{h}^0 \subseteq (\sigma - \tilde\sigma)\mathfrak{h}^0 \quad \text{and} \quad (\sigma - \tilde\sigma)\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0.$$

Thus for the kernel of $\widehat{\zeta}$, we have the equivalences

$$(\sigma - \tilde\sigma)w \in \ker\widehat{\zeta} \iff \partial w \in \ker\widehat{\zeta}$$
$$\iff \forall n \in \mathbf{Z}^+,\ \widehat{\zeta}[\partial_n w] = 0.$$

□