

# On $q$ -Analog of Multiple Zeta Values and other Multiple Harmonic Series

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August 17, 2004

## Multiple Harmonic Sums

Example:

$$\begin{aligned} Z_n^{\geq}(s, t, u) &:= \sum_{i=1}^n \frac{1}{i^s} \sum_{j=1}^i \frac{1}{j^t} \sum_{k=1}^j \frac{1}{k^u} \\ &= \sum_{n \geq i \geq j \geq k \geq 1} i^{-s} j^{-t} k^{-u} \end{aligned}$$

- $i \geq j \geq k$  are positive integers
- $n$  may be finite or infinite ( $0 \leq n \leq \infty$ )
- $s, t, u$  are positive integers ( $s, t, u \in \mathbf{Z}^+$ )
- $s > 1$  if  $n = \infty$

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With strict inequalities:

$$\begin{aligned} Z_n^>(s, t, u) &:= \sum_{i=1}^n \frac{1}{i^s} \sum_{j=1}^{i-1} \frac{1}{j^t} \sum_{k=1}^{j-1} \frac{1}{k^u} \\ &= \sum_{n \geq i > j > k \geq 1} i^{-s} j^{-t} k^{-u} \end{aligned}$$

- $i > j > k$  are positive integers
- $n$  may be finite or infinite ( $0 \leq n \leq \infty$ )
- $s, t, u$  are positive integers ( $s, t, u \in \mathbf{Z}^+$ )
- $s > 1$  if  $n = \infty$

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$$\begin{aligned} Z_n^{\geq}(s_1, \dots, s_m) &:= \sum_{k_1=1}^n \frac{1}{k_1^{s_1}} \sum_{k_2=1}^{k_1} \frac{1}{k_2^{s_2}} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{1}{k_m^{s_m}} \\ &= \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m k_j^{-s_j} \end{aligned}$$

- $k_1 \geq k_2 \geq \dots \geq k_m$  are positive integers
- $n$  may be finite or infinite ( $0 \leq n \leq \infty$ )
- $s_1, \dots, s_m$  are positive integers ( $\forall j, s_j \in \mathbf{Z}^+$ )
- $s_1 > 1$  if  $n = \infty$

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$$Z_n^>(s_1, \dots, s_m) := \sum_{k_1=1}^n \frac{1}{k_1^{s_1}} \sum_{k_2=1}^{k_1-1} \frac{1}{k_2^{s_2}} \cdots \sum_{k_m=1}^{k_{m-1}-1} \frac{1}{k_m^{s_m}}$$

$$= \sum_{n \geq k_1 > k_2 > \cdots > k_m \geq 1} \prod_{j=1}^m k_j^{-s_j}$$

- $k_1 > k_2 > \cdots > k_m$  are positive integers
- $n$  may be finite or infinite ( $0 \leq n \leq \infty$ )
- $s_1, \dots, s_m$  are positive integers ( $\forall j, s_j \in \mathbf{Z}^+$ )
- $s_1 > 1$  if  $n = \infty$
- $\zeta(s_1, \dots, s_m) := Z_\infty^>(s_1, \dots, s_m)$  is called a *multiple zeta value* (MZV)

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## Relationship between $Z_n^>$ and $Z_n^\geq$

$$Z_n^>(s) = Z_n^\geq(s),$$

$$Z_n^>(s, t) = Z_n^\geq(s, t) + Z_n^>(s + t),$$

$$Z_n^\geq(s, t, u) = Z_n^\geq(s, t, u) + Z_n^\geq(s + t, u)$$

$$+ Z_n^\geq(s, t + u) + Z_n^\geq(s + t + u).$$

More generally, let  $\vec{s} = (s_1, s_2, \dots, s_m)$ . Then

$$Z_n^\geq(\vec{s}) = \sum Z_n^\geq(\vec{t}),$$

where the sum is over all  $\vec{t}$  obtained from  $\vec{s}$  by replacing any number of commas by plus signs.

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## Positive $q$ -Integers

**Definition 1** The  $q$ -analog of  $k \in \mathbf{Z}^+$  is

$$[k]_q := \sum_{j=0}^{k-1} q^j$$

$$= 1 + q + q^2 + \cdots + q^{k-1}$$

$$= \begin{cases} \frac{1 - q^k}{1 - q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

Note that  $\lim_{q \rightarrow 1} [k]_q = k$ .

Can we find reasonable/interesting  $q$ -analogs of multiple harmonic sums?

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## $q$ -Powers and $q$ -Binomial Coefficients

Let  $0 \leq n \in \mathbf{Z}$  and  $x, y \in \mathbf{R}$ .

Define the asymmetric  $q$ -power by

$$(x + y)_q^n := \prod_{k=0}^{n-1} (x + yq^k).$$

Clearly,

$$\lim_{q \rightarrow 1} (x + y)_q^n = (x + y)^n.$$

The Gaussian binomial coefficient (a.k.a. the  $q$ -binomial coefficient) is defined for  $0 \leq n \in \mathbf{Z}$  and integer  $k$  by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \frac{(1 - q)_q^n}{(1 - q)_q^k (1 - q)_q^{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

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## The $q$ -Factorial

If we set

$$[n]!_q := \prod_{k=1}^n [k]_q = \prod_{k=1}^n \frac{1-q^k}{1-q} = \frac{(1-q)_q^n}{(1-q)^n},$$

then evidently

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

and thus

$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

We also have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^k \frac{1-q^{n-k+j}}{1-q^j}, \quad 0 \leq k \leq n,$$

$$[n]!_q = \sum_{k=0}^{n(n-1)/2} a_k q^k, \\ a_k = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } k \text{ inversions}\}.$$

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## The $q$ -Binomial Theorem

**Theorem 2 (Finite  $q$ -Binomial Theorem)** Let  $x, y \in \mathbb{R}$  and  $0 \leq n \in \mathbb{Z}$ . Then

$$(x+y)_q^n = \sum_{k=0}^n q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k.$$

Letting  $q \rightarrow 1$ , we deduce

**Corollary 1 (Classical Binomial Theorem)**

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

**Corollary 2 ( $q$ -Vandermonde Convolution)**

$$\begin{bmatrix} a+b \\ n \end{bmatrix} = \sum_{k=0}^n q^{(n-k)(a-k)} \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ n-k \end{bmatrix}.$$

**Proof Sketch.** Compare coefficients of  $y^n$  in

$$(1+y)_q^{a+b} = (1+y)_q^a (1+yq^a)_q^b.$$

□

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**Theorem 3 (G.-C. Rota)** Let  $q$  be a power of a prime positive integer. Then  $\begin{bmatrix} n \\ k \end{bmatrix}$  is equal to the number of  $k$ -dimensional subspaces of the  $n$ -dimensional vector space

$$\mathbf{F}_q^{\oplus n} = \underbrace{\mathbf{F}_q \oplus \cdots \oplus \mathbf{F}_q}_n,$$

where  $\mathbf{F}_q$  is the finite field with  $q$  elements.

**Proof (Sketch).** Fix  $1 \leq k \leq n$ . Any  $k$ -dimensional subspace is determined by a basis consisting of  $k$  linearly independent spanning vectors.

There are  $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$  bases for all  $k$ -dimensional subspaces.

Since different bases may span the same subspace, we divide by the number of possible choices of basis of a particular  $k$ -dimensional subspace:

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}).$$

The quotient is  $\begin{bmatrix} n \\ k \end{bmatrix}$ . □

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## The $q$ -Exponential Function

If  $0 < q < 1$ , then

$$\lim_{n \rightarrow \infty} \begin{bmatrix} n \\ k \end{bmatrix} = \lim_{n \rightarrow \infty} \prod_{j=1}^k \frac{1-q^{n-k+j}}{1-q^j} = \prod_{j=1}^k \frac{1}{1-q^j} \\ = \frac{1}{(1-q)_q^k [k]!_q}.$$

Recall the  $q$ -binomial theorem in the form

$$(1+x)_q^n = \sum_{k=0}^n q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad 0 \leq n \in \mathbb{Z}.$$

Thus,

$$\prod_{j=0}^{\infty} (1+xq^j) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{[k]!_q} \left(\frac{x}{1-q}\right)^k.$$

Define the  $q$ -exponential function by

$$\exp_q(x) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{[k]!_q} = (1+(1-q)x)_q^{\infty}.$$

Then

$$\lim_{q \rightarrow 1} \exp_q(x) = e^x, \quad D_q \exp_q(x) = \exp_q(qx).$$

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## The $q$ -Gamma Function

The  $q$ -gamma function has infinite product representation

$$\begin{aligned}\Gamma_q(t) &:= \frac{(1-q)_q^\infty}{(1-q)^{t-1}(1-q^t)_q^\infty} \\ &= (1-q)^{1-t} \prod_{k=1}^{\infty} \frac{1-q^k}{1-q^{t+k-1}}.\end{aligned}$$

For  $t > 0$ , we have also

$$\Gamma_q(t) := \int_0^\infty x^{t-1} \exp_q(-qx) d_q x.$$

It can be shown that  $\lim_{q \rightarrow 1} \Gamma_q(t) = \Gamma(t)$ .

Also  $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$ ,  $\Gamma_q(1) = 1$ , so

$$\Gamma_q(n+1) = [n]_q!, \quad 0 \leq n \in \mathbf{Z}.$$

Askey proved a  $q$ -analog of the Bohr-Mollerup theorem: If  $f(1) = 1$ ,  $f(t+1) = [t]_q f(t)$  for some  $0 < q < 1$ , and  $\log f$  is convex for  $x > 0$ , then  $f(x) = \Gamma_q(x)$ .

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## $q$ -Harmonic Sums

Suppose we define

$$Z_n[s] := \sum_{n \geq k \geq 1} \frac{1}{[k]_q^s} = (1-q)^s \sum_{k=1}^n \frac{1}{(1-q^k)^s}.$$

If  $0 < q < 1$ , then  $\lim_{k \rightarrow \infty} \frac{1}{(1-q^k)^s} = 1$ .

Consequently,  $Z_\infty[s] = \sum_{k=1}^{\infty} \frac{1}{[k]_q^s}$  diverges.

### Options:

- Restrict  $n$  to be finite ( $0 \leq n < \infty$ )
- Insist that  $q > 1$  if  $n = \infty$
- Insert convergence factors into the sum

$$\text{eg. } \sum_{k=1}^n \frac{q^k}{[k]_q^s} \text{ or } \sum_{k=1}^n \frac{q^{sk}}{[k]_q^s} \text{ or } \sum_{k=1}^n \frac{q^{(s-1)k}}{[k]_q^s}$$

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L. Van Hamme:

$$\sum_{k=1}^n \frac{q^k}{1-q^k} = \sum_{k=1}^n \frac{(-1)^{k+1} q^{k(k+1)/2}}{1-q^k} \left[ \begin{matrix} n \\ k \end{matrix} \right].$$

K. Dilcher:

$$\begin{aligned}\sum_{k_1=1}^n \frac{q^{k_1}}{1-q^{k_1}} \sum_{k_2=1}^{k_1} \frac{q^{k_2}}{1-q^{k_2}} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m}}{1-q^{k_m}} \\ = \sum_{k=1}^n \frac{(-1)^{k+1} q^{k(k+1)/2 + (m-1)k}}{(1-q^k)^m} \left[ \begin{matrix} n \\ k \end{matrix} \right].\end{aligned}$$

Equivalently,

$$\begin{aligned}\sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q} \\ = \sum_{k=1}^n \frac{(-1)^{k+1} q^{k(k+1)/2 + (m-1)k}}{[k]_q^m} \left[ \begin{matrix} n \\ k \end{matrix} \right].\end{aligned}$$

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## Multiple $q$ -Harmonic Sums

**Definition 4** Let  $n, m$  and  $s_1, s_2, \dots, s_m$  be positive integers. Define

$$Z_n[s_1, \dots, s_m] := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}},$$

$$\begin{aligned}A_n[s_1, \dots, s_m] := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} (-1)^{k_1+1} \\ \times q^{k_1(k_1+1)/2} \left[ \begin{matrix} n \\ k_1 \end{matrix} \right] \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.\end{aligned}$$

Both sums vanish if  $n = 0$ . If  $n > 0$  and  $m = 0$ , define  $Z_n[] = A_n[] = 1$ .

### Abbreviations:

$\text{Cat}_{j=1}^m \{s_j\}$  denotes the sequence  $s_1, s_2, \dots, s_m$ .

$$\{s\}^m := \text{Cat}_{j=1}^m \{s\} = \underbrace{s, \dots, s}_m$$

(i.e.  $m \geq 0$  consecutive copies of  $s$ ).

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**Theorem 5** Let  $n, r, a_1, b_1, \dots, a_r, b_r \in \mathbf{Z}^+$ . Then

$$\begin{aligned} Z_n \left[ \text{Cat}_{j=1}^{r-1} \{ \{1\}^{a_j-1}, b_j + 1 \}, \{1\}^{a_r-1}, b_r \right] \\ = A_n \left[ a_1, \{1\}^{b_1-1}, \text{Cat}_{j=2}^r \{ a_j + 1, \{1\}^{b_j-1} \} \right]. \end{aligned}$$

**Example 1** Putting  $r = 2$ ,  $a_1 = 3$ ,  $b_1 = 2$ ,  $a_2 = b_2 = 1$  gives  $Z_n[1, 1, 3, 1] = A_n[3, 1, 2]$ , i.e.

$$\begin{aligned} \sum_{n \geq j \geq k \geq m \geq p \geq 1} \frac{q^{j+k+m+p}}{[j]_q [k]_q [m]_q^3 [p]_q} \\ = \sum_{n \geq k \geq m \geq p \geq 1} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{2k+p}}{[k]_q^3 [m]_q [p]_q^2}. \end{aligned}$$

**Example 2** Putting  $r = 2$ ,  $a_1 = a_2 = b_1 = 1$ ,  $b_2 = 2$  in Theorem 5 gives  $Z_n[2, 2] = A_n[1, 2, 1]$ , i.e.

$$\begin{aligned} \sum_{n \geq k \geq m \geq 1} \frac{q^{k+m}}{[k]_q^2 [m]_q^2} \\ = \sum_{n \geq k \geq m \geq p \geq 1} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^m}{[k]_q [m]_q^2 [p]_q}. \end{aligned}$$

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## $A_n/Z_n$ Duality I

Define an involution on the set  $\mathcal{S}$  of finite sequences of positive integers as follows:

Let  $\alpha$  be the map that sends a sequence in  $\mathcal{S}$  to its sequence of partial sums.

Let  $\beta$  be the involution on strictly increasing sequences in  $\mathcal{S}$  defined by

$$\beta(a_1, \dots, a_k) = \{1, 2, 3, \dots, a_k\} \setminus \{a_1, \dots, a_{k-1}\}$$

arranged in increasing order.

Theorem 5 can now be restated as

$$Z_n[\vec{s}] = A_n[\alpha^{-1}\beta\alpha\vec{s}], \quad \forall \vec{s} \in \mathcal{S}, 0 < n \leq \infty.$$

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## $A_n/Z_n$ Duality II

Let  $\mathfrak{h} = \mathbf{Q}\langle x, y \rangle$  and  $\mathfrak{h}' = \mathbf{Q}1 \oplus \mathfrak{h}y$  and fix  $0 < n \leq \infty$ .

Define  $\mathbf{Q}$ -linear maps  $\widehat{A}_n$  and  $\widehat{Z}_n$  on  $\mathfrak{h}'$  by

$$\begin{aligned} \widehat{A}_n[1] &:= A_n[ ] = 1, \\ \widehat{A}_n \left[ \prod_{j=1}^k x^{s_j-1} y \right] &:= A_n[s_1, \dots, s_k], \quad s_j \in \mathbf{Z}^+, \end{aligned}$$

and similarly for  $\widehat{Z}_n$ .

Let  $J$  be the automorphism of  $\mathfrak{h}$  that switches  $x$  and  $y$ .

Define an involution of  $\mathfrak{h}'$  by

$$w^* = (Jw)x^{-1}y, \quad \forall w \in \mathfrak{h}'.$$

Then Theorem 5 can be restated as

$$\widehat{A}_n[w] = \widehat{Z}_n[w^*], \quad \forall w \in \mathfrak{h}'.$$

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Recall Theorem 5:

$$\begin{aligned} Z_n \left[ \text{Cat}_{j=1}^{r-1} \{ \{1\}^{a_j-1}, b_j + 1 \}, \{1\}^{a_r-1}, b_r \right] \\ = A_n \left[ a_1, \{1\}^{b_1-1}, \text{Cat}_{j=2}^r \{ a_j + 1, \{1\}^{b_j-1} \} \right]. \end{aligned}$$

**Corollary 3** Let  $n, a, b \in \mathbf{Z}^+$ . Then

$$Z_n[\{1\}^{a-1}, b] = A_n[a, \{1\}^{b-1}].$$

**Proof.** Put  $r = 1$  in Theorem 5. □

**Example 3** Putting  $b = 1$  and  $a = m$  yields

$$Z_n[\{1\}^m] = A_n[m],$$

which is Dilcher's result

$$\begin{aligned} \sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q} \\ = \sum_{k=1}^n (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{(m-1)k}}{[k]_q^m}. \end{aligned}$$

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Note that Dilcher's result

$$\sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q} = \sum_{k=1}^n (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{(m-1)k}}{[k]_q^m}.$$

has limiting cases

$$\sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{1}{k_j} = \sum_{k=1}^n \frac{(-1)^{k+1}}{k^m} \binom{n}{k},$$

and

$$\sum_{k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{1}{k_j} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} q^{k(k+1)/2 + (m-1)k}}{(1-q)_q^k [k]_q^m}.$$

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Recall Corollary 3: If  $n, a, b \in \mathbf{Z}^+$  then

$$Z_n[\{1\}^{a-1}, b] = A_n[a, \{1\}^{b-1}].$$

**Example 4** Putting  $a = 1$  and  $b = m$  yields

$$Z_n[m] = A_n[\{1\}^m],$$

i.e.

$$\sum_{k=1}^n \frac{q^k}{[k]_q^m} = \sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} (-1)^{k_1+1} q^{k_1(k_1+1)/2} \begin{bmatrix} n \\ k_1 \end{bmatrix} \prod_{j=1}^m \frac{1}{[k_j]_q},$$

with limiting cases

$$\sum_{k=1}^n \frac{1}{k^m} = \sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} (-1)^{k_1+1} \binom{n}{k_1} \prod_{j=1}^m \frac{1}{k_j}$$

and

$$\sum_{k=1}^{\infty} \frac{q^k}{[k]_q^m} = \sum_{k_1 \geq \dots \geq k_m \geq 1} \frac{(-1)^{k_1+1} q^{k_1(k_1+1)/2}}{(1-q)_q^{k_1}} \prod_{j=1}^m \frac{1}{[k_j]_q}.$$

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## Proof of Theorem 5

By induction, it suffices to establish the base cases

$$A_n[\ ] = A_n[0] = 1 \text{ for } 0 < n \in \mathbf{Z}$$

and the following two recurrence relations:

**Proposition 6** Let  $n, m$  and  $s_1, s_2, \dots, s_m \in \mathbf{Z}^+$ . Then

$$A_n[s_1, \dots, s_m] = \sum_{r=1}^n \frac{q^r}{[r]_q} A_r[s_1 - 1, s_2, \dots, s_m].$$

**Proposition 7** Let  $n, m$  and  $s_2, s_3, \dots, s_m \in \mathbf{Z}^+$ . Then

$$A_n[0, s_2, s_3, \dots, s_m] = \frac{A_n[s_2 - 1, s_3, \dots, s_m]}{[n]_q}.$$

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## Base Cases

The base case

$$A_n[\ ] = 1 \text{ for } n > 0$$

is true by definition.

The other base case, namely

$$A_n[0] = 1 \text{ for } n > 0$$

is an easy consequence of the  $q$ -binomial theorem:

$$(x+y)_q^n = \sum_{k=0}^n q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k.$$

If  $n > 0$ , Putting  $x = 1$  and  $y = -1$  gives

$$\begin{aligned} A_n[0] &= \sum_{k=1}^n (-1)^{k+1} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \\ &= 1 - (1-1)_q^n \\ &= 1. \end{aligned}$$

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## Proof of Proposition 6 (1st recurrence for $A_n$ )

We begin with the  $q$ -Pascal recurrence

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k \end{bmatrix} + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}.$$

**Lemma 8** Let  $k$  and  $n$  be positive integers with  $k \leq n$ . Then

$$\sum_{r=k}^n q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = q^k \begin{bmatrix} n \\ k \end{bmatrix}.$$

**Proof.** Write the  $q$ -Pascal recurrence in the form

$$q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} r \\ k \end{bmatrix} - \begin{bmatrix} r-1 \\ k \end{bmatrix},$$

multiply through by  $q^k$ , and sum on  $r$ .  $\square$

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**Definition 9** Let  $n, m$ , and  $s_1, \dots, s_m$  be non-negative integers. Define

$$W_n[s_1, \dots, s_m] := \sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

If  $n = 0$ , the sum is empty and  $W_0[\vec{s}] = 0$ . If  $n > 0$  and  $m = 0$ , define  $W_n[] := 1$ .

We can now write

$$\begin{aligned} A_n[s_1, \dots, s_m] &= \sum_{k=1}^n (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{(s_1-1)k}}{[k]_q^{s_1}} W_k[s_2, \dots, s_m] \\ &= \sum_{k=1}^n \frac{(-1)^{k+1} q^{k(k-1)/2 + (s_1-1)k}}{[k]_q^{s_1}} W_k \boxed{q^k \begin{bmatrix} n \\ k \end{bmatrix}}. \end{aligned}$$

By Lemma 8, the boxed expression is  $\sum_{r=k}^n q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}$ .

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It follows that

$$\begin{aligned} A_n[s_1, \dots, s_m] &= \sum_{k=1}^n \frac{(-1)^{k+1} q^{k(k-1)/2 + (s_1-1)k}}{[k]_q^{s_1}} W_k \sum_{r=k}^n q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \\ &= \sum_{r=1}^n q^r \sum_{k=1}^r \frac{(-1)^{k+1} q^{k(k-1)/2 + (s_1-1)k}}{[k]_q^{s_1}} W_k \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \\ &= \sum_{r=1}^n \frac{q^r}{[r]_q} \sum_{k=1}^r \frac{(-1)^{k+1} q^{k(k+1)/2 + (s_1-2)k}}{[k]_q^{s_1-1}} W_k \begin{bmatrix} r \\ k \end{bmatrix} \\ &= \sum_{r=1}^n \frac{q^r}{[r]_q} A_r[s_1-1, s_2, \dots, s_m]. \end{aligned}$$

$\square$

In the penultimate step, we used

$$\frac{[r]_q}{[k]_q} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} r \\ k \end{bmatrix}, \quad 1 \leq k \leq r.$$

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## Multiple Zeta Values

$$\zeta(s_1, \dots, s_m) := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m k_j^{-s_j}.$$

The multiple series is absolutely convergent if

$$\sum_{j=1}^n \Re(s_j) > n, \quad n = 1, 2, \dots, m.$$

Euler ( $m = 2$ ):

$$2\zeta(s, 1) = s\zeta(s+1) - \sum_{j=1}^{s-2} \zeta(s-j)\zeta(j+1),$$

where  $2 \leq s \in \mathbf{Z}$ .

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## Period One

For all non-negative integers  $n$ ,

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\{4\}^n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!},$$

$$\zeta(\{6\}^n) = \frac{6 \cdot (2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(\{8\}^n) = \frac{8 \cdot (2\pi)^{8n}}{(8n+4)!} \times \left\{ \left(1 + \frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1 - \frac{1}{\sqrt{2}}\right)^{4n+2} \right\}.$$

More generally, let  $k \in \mathbf{Z}^+$  and  $\omega := e^{i\pi/k}$ . Then

$$\sum_{n=0}^{\infty} (-1)^n x^{2kn} \zeta(\{2k\}^n) = \prod_{j=0}^{k-1} \frac{\sin(\pi x \omega^j)}{\pi x \omega^j}.$$

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## Period Two

For all non-negative integers  $n$ ,

$$\zeta(\{3, 1\}^n) = 4^{-n} \zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!},$$

$$\begin{aligned} \zeta(3, \{1, 3\}^n) &= 4^{-n} \sum_{k=0}^n \zeta(4k+3) \zeta(\{4\}^{n-k}) \\ &= \sum_{k=0}^n \frac{2\pi^{4k}}{(4k+2)!} \left(-\frac{1}{4}\right)^{n-k} \zeta(4n-4k+3), \end{aligned}$$

$$\begin{aligned} \zeta(2, \{1, 3\}^n) &= 4^{-n} \sum_{k=0}^n (-1)^k \zeta(\{4\}^{n-k}) \left\{ (4k+1) \right. \\ &\quad \left. \times \zeta(4k+2) - 4 \sum_{j=1}^k \zeta(4j-1) \zeta(4k-4j+3) \right\}. \end{aligned}$$

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The study of multiple zeta values touches on

- Number theory: algebraic independence
- Knot theory: polynomials of Kaufman, Kontsevich
- Quantum field theory: Feynman diagrams
- Algebra: Quasi-symmetric functions,  $K$ -theory
- Combinatorics: shuffles, quad-trees
- Symbolic computation: Gröbner bases
- Numerical computation: hunting for or disproving the existence of identities

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- Partial fractions: Euler, Nielsen, Y. Ohno
- Series transformations: Subbarao, Ramanujan
- Linear algebra: D. Bailey, R. Girgensohn
- Contour integration: P. Flajolet & B. Salvy
- Special functions: J. & D. Borwein, DB
- Differential equations: D. Bowman, DB
- Harmonic algebra: M. Hoffman
- Lyndon Bases: D. Broadhurst, M. Bigotte
- Analytic Continuation: T. Arakawa, M. Kaneko, S. Akiyama, S. Egami, Y. Tanigawa, J. Zhao
- Combinatorics: H. Minh, J. Ryoo, DB
- Tate Motives: T. Terasoma, A. Goncharov
- Knot Theory: D. Kreimer, T. Le, J. Murakami

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## Multiple $q$ -Zeta Values

In analogy with

$$Z_n^{\geq}[s_1, \dots, s_m] = \sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}},$$

define

$$Z_n^>[s_1, \dots, s_m] = \sum_{n \geq k_1 > \dots > k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

Then

$$Z_{\infty}^>[s_1, \dots, s_m] = \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}$$

is a  $q$ -analog of the multiple zeta value

$$\zeta(s_1, \dots, s_m) = \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{1}{k_j^{s_j}}.$$

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M. Kaneko investigated analytic properties of the Riemann  $q$ -zeta function

$$\zeta[s] := \sum_{k=1}^{\infty} \frac{q^{tk}}{[k]_q^s}, \quad t = s - 1.$$

This suggests we consider

$$\zeta[s_1, \dots, s_m] := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Recall we previously defined

$$Z_{\infty}^>[s_1, \dots, s_m] = \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

Let  $\vec{s} = (s_1, \dots, s_m)$ . Then

$$\zeta[\vec{s}; q] = q^{|\vec{s}|} Z_{\infty}^>[\vec{s}; 1/q], \quad |\vec{s}| := \sum_{j=1}^m s_j.$$

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## Stuffle Multiplication

**Example.**

$$\begin{aligned} \zeta(a)\zeta(b, c) &= \sum_{n>0} n^{-a} \sum_{m>k>0} m^{-b} k^{-c} \\ &= \sum_{n>m>k>0} n^{-a} m^{-b} k^{-c} \\ &\quad + \sum_{n=m>k>0} n^{-a-b} k^{-c} \\ &\quad + \sum_{m>n>k>0} m^{-b} n^{-a} k^{-c} \\ &\quad + \sum_{m>n=k>0} m^{-b} k^{-a-c} \\ &\quad + \sum_{m>k>n>0} m^{-b} k^{-c} n^{-a} \\ &= \zeta(a, b, c) + \zeta(a+b, c) + \zeta(b, a, c) \\ &\quad + \zeta(b, a+c) + \zeta(b, c, a). \end{aligned}$$

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Let  $u$  and  $v$  be (ordered) lists of positive integers. Then

$$\zeta(u)\zeta(v) = \sum_{w \in u*v} \zeta(w),$$

where  $u*v$  is the multiset defined by the recursion

$$\begin{aligned} (s, u) * (t, v) &= (s, u * (t, v)) \cup (t, (s, u) * v) \cup (s+t, u * v), \\ \varepsilon * u &= u * \varepsilon = u, \end{aligned}$$

where  $\varepsilon$  is the empty list.

eg.

$$\begin{aligned} a * (b, c) &= (a, \varepsilon * (b, c)) \cup (b, a * c) \cup (a+b, \varepsilon * c) \\ &= \{(a, (b, c)), (b, a, \varepsilon * c), (b, c, a * \varepsilon), \\ &\quad (b, a+c, \varepsilon * \varepsilon), (a+b, c)\} \\ &= \{(a, b, c), (b, a, c), (b, c, a), \\ &\quad (b, a+c), (a+b, c)\}, \end{aligned}$$

whence

$$\begin{aligned} \zeta(a)\zeta(b, c) &= \zeta(a, b, c) + \zeta(b, a, c) + \zeta(b, c, a) \\ &\quad + \zeta(b, a+c) + \zeta(a+b, c). \end{aligned}$$

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## Counting Stuffles

Let  $f(|u|, |v|)$  denote the number of lists in  $u * v$ .

The recursive decomposition implies that the generating function

$$F(x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) x^m y^n$$

satisfies the functional equation

$$F(x, y) = 1 + xF(x, y) + yF(x, y) + xyF(x, y).$$

It follows that

$$F(x, y) = (1 - x - y - xy)^{-1},$$

$$\begin{aligned} f(m, n) &= \sum_{k=0}^m \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^{\min(m, n)} \binom{n}{k} \binom{m}{k} 2^k \\ &= \left| \left\{ (b_1, \dots, b_m) \in \mathbf{Z}^m : \sum_{j=1}^m |b_j| \leq n \right\} \right|. \end{aligned}$$

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## Explicit Description

Abbreviate  $\{1, 2, \dots, n\}$  by  $\langle n \rangle$ .

Define a *stuffle* on  $(m, n) \in \mathbf{Z}^+ \times \mathbf{Z}^+$  as a pair  $(\phi, \psi)$  of order-preserving injective mappings

$$\phi : \langle m \rangle \rightarrow \langle m+n \rangle, \quad \psi : \langle n \rangle \rightarrow \langle m+n \rangle$$

such that the union of their images is equal to  $\langle r \rangle$  for some  $r$  with  $\max(m, n) \leq r \leq m+n$ .

Stuffle multiplication can now be written in the form

$$\begin{aligned} \zeta(s_1, \dots, s_m) \zeta(t_1, \dots, t_n) \\ = \sum_{(\phi, \psi)} \zeta \left( \mathbf{Cat}_{k=1}^r \{s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)}\} \right), \end{aligned}$$

where the sum is over all stuffles  $(\phi, \psi)$  on  $(m, n)$ , and  $r = r(\phi, \psi)$  is as above.

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$$\begin{aligned} \frac{q^{(s+t-2)k}}{[k]_q^{s+t}} &= \frac{q^{(s+t-1)k}}{[k]_q^{s+t}} \left\{ 1 + \frac{1}{q^k} - 1 \right\} \\ &= \frac{q^{(s+t-1)k}}{[k]_q^{s+t}} + \frac{q^{(s+t-1)k}}{[k]_q^{s+t-1}} \cdot \frac{(1-q^k)}{q^k} \\ &= \frac{q^{(s+t-1)k}}{[k]_q^{s+t}} + (1-q) \frac{q^{(s+t-2)k}}{[k]_q^{s+t-1}}. \end{aligned}$$

$$\begin{aligned} \zeta[s] \zeta[t] &= \sum_{k>0} \frac{q^{(s-1)k}}{[k]_q} \sum_{j>0} \frac{q^{(t-1)j}}{[j]_q} \\ &= \sum_{k>j>0} \frac{q^{(s-1)k+(t-1)j}}{[k]_q^s [j]_q^t} \\ &\quad + \sum_{j>k>0} \frac{q^{(t-1)j+(s-1)k}}{[j]_q^t [k]_q^s} + \sum_{k>0} \frac{q^{(s+t-2)k}}{[k]_q^{s+t}} \\ &= \zeta[s, t] + \zeta[t, s] + \zeta[s+t] \\ &\quad + (1-q) \zeta[s+t-1]. \end{aligned}$$

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## $q$ -Stuffles

More generally, expanding the product

$$\zeta[s_1, \dots, s_m] \zeta[t_1, \dots, t_n]$$

yields sums of products of terms of the form

$$\frac{q^{(s-1)k+(t-1)j}}{[k]_q^s [j]_q^t},$$

which, if  $k = j$ , reduces to

$$\frac{q^{(s+t-2)k}}{[k]_q^{s+t}} = (1-q) \frac{q^{(s+t-2)k}}{[k]_q^{s+t-1}} + \frac{q^{(s+t-1)k}}{[k]_q^{s+t}}.$$

It follows that

$$\begin{aligned} \zeta[s_1, \dots, s_m] \zeta[t_1, \dots, t_n] \\ = \sum_{(\phi, \psi)} \sum_A (1-q)^{|A|} \zeta \left[ \mathbf{Cat}_{k=1}^r \{s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)} - (k \in A)\} \right], \end{aligned}$$

where the outer sum is over all stuffles  $(\phi, \psi)$  on  $(m, n)$ , the inner sum is over all subsets  $A$  of the intersection of the images of  $\phi$  and  $\psi$ ,  $r = |\phi(\langle m \rangle) \cup \psi(\langle n \rangle)|$  as before, and the Boolean expression  $(k \in A)$  takes the value 1 if true (i.e.  $k \in A$ ) and 0 if false (i.e.  $k \notin A$ ).

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## Period-1 Sums Reduce

**Theorem 10** If  $n$  is a positive integer and  $s > 1$ , then

$$n\zeta[\{s\}^n] = \sum_{k=1}^n (-1)^{k+1} \zeta[\{s\}^{n-k}] \sum_{j=0}^{k-1} \binom{k-1}{j} (1-q)^j \zeta[ks-j].$$

**Example 5** With  $n = 2$ , we get

$$2\zeta[s, s] = \zeta[s]\zeta[s] - (\zeta[2s] + (1-q)\zeta[2s-1]).$$

**Corollary 4** If  $n$  is a positive integer and  $s > 1$ , then

$$n\zeta(\{s\}^n) = \sum_{k=1}^n (-1)^{k+1} \zeta(\{s\}^{n-k}) \zeta(ks).$$

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Let  $\mathfrak{S}_n$  denote the group of  $n!$  permutations of  $\langle n \rangle = \{1, 2, \dots, n\}$ .

**Theorem 11** Let  $n$  be a positive integer, and let  $s_j > 1$  for  $1 \leq j \leq n$ . Then

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta \left[ \text{Cat}_{j=1}^n s_{\sigma(j)} \right] = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{k=1}^{|\mathcal{P}|} (|P_k| - 1)! \times \sum_{\nu_k=0}^{|P_k|-1} \binom{|P_k|-1}{\nu_k} (1-q)^{\nu_k} \zeta[p_k - \nu_k],$$

where the outer sum on the right is over all unordered set partitions  $\mathcal{P} = \{P_1, \dots, P_m\}$  of  $\langle n \rangle$ ,  $1 \leq m = |\mathcal{P}| \leq n$ , and  $p_k = \sum_{j \in P_k} s_j$ .

**Corollary 5 (M. Hoffman)**

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta \left( \text{Cat}_{j=1}^n s_{\sigma(j)} \right) = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{P \in \mathcal{P}} (|P| - 1)! \zeta \left( \sum_{j \in P} s_j \right).$$

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## Parity Reduction

**Theorem 12** Let  $m \in \mathbf{Z}^+$  and let  $s_1, \dots, s_m$  be real numbers with  $s_1 > 1$ ,  $s_m > 1$ , and  $s_j \geq 1$  for  $1 < j < m$ . Then

$$\zeta \left[ \text{Cat}_{k=1}^m s_k \right] + (-1)^m \zeta \left[ \text{Cat}_{k=1}^m s_{m-k+1} \right]$$

can be expressed as a  $\mathbf{Z}[q]$ -linear combination of multiple  $q$ -zeta values of depth less than  $m$ .

That is, the coefficients in the linear combination are polynomials in  $q$  with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

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## A Double Generating Function

**Theorem 13**

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} \zeta[m+2, \{1\}^n] = 1 - \exp \left\{ \sum_{k=2}^{\infty} \left\{ u^k + v^k - (u+v+(1-q)uv)^k \right\} \times \frac{1}{k} \sum_{j=2}^k (q-1)^{k-j} \zeta[j] \right\}.$$

**Corollary 6** If  $0 \leq m, n \in \mathbf{Z}$ , then

$$\zeta[m+2, \{1\}^n] = \zeta[n+2, \{1\}^m].$$

**Corollary 7 (q-Euler)** Let  $0 \leq m \in \mathbf{Z}$ . Then

$$2\zeta[m+2, 1] = (m+2)\zeta[m+3] + (1-q)m\zeta[m+2] - \sum_{k=2}^{m+1} \zeta[m+3-k] \zeta[k].$$

In particular, when  $m = 0$  we get  $\zeta[2, 1] = \zeta[3]$ .

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The proof of Theorem 13 makes essential use of the basic hypergeometric function

$${}_2\phi_1 \left[ \begin{matrix} q^a, q^b \\ q^c \end{matrix} \middle| x \right] = \sum_{n=0}^{\infty} \frac{(1-q^a)_q^n (1-q^b)_q^n}{(1-q^c)_q^n (1-q)_q^n} x^n, \quad |x| < 1.$$

Routine series manipulations reveal that

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} [x]_q^{m+1} [y]_q^{n+1} \zeta[m+2, \{1\}^n] \\ = 1 - {}_2\phi_1 \left[ \begin{matrix} q^{-y}, q^x \\ q^{1+x} \end{matrix} \middle| q^{1+y} \right] \end{aligned}$$

Heine's  $q$ -analog

$${}_2\phi_1 \left[ \begin{matrix} q^a, q^b \\ q^c \end{matrix} \middle| q^{c-a-b} \right] = \frac{\Gamma_q(c) \Gamma_q(c-a-b)}{\Gamma_q(c-a) \Gamma_q(c-b)}$$

of Gauss's  ${}_2F_1$  summation formula then gives

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} [x]_q^{m+1} [y]_q^{n+1} \zeta[m+2, \{1\}^n] \\ = 1 - \frac{\Gamma_q(1+x) \Gamma_q(1+y)}{\Gamma_q(1+x+y)}. \end{aligned}$$

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## The Simplex Integral

M. Kontsevich: If  $s_1, \dots, s_m \in \mathbf{Z}^+$ , then

$$\zeta(s_1, \dots, s_m) = \int \prod_{k=1}^m \left( \prod_{r=1}^{s_k-1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1-t_{s_k}^{(k)}}$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \dots > t_{s_1}^{(1)} > \dots > t_1^{(m)} > \dots > t_{s_m}^{(m)} > 0,$$

and is abbreviated (D. Broadhurst) by

$$\int_0^1 \prod_{k=1}^m A^{s_k-1} B, \quad A = \frac{dt}{t}, \quad B = \frac{dt}{1-t}.$$

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If  $\vec{s} = (s_1, \dots, s_m)$  define

$$\text{weight}(\vec{s}) := |\vec{s}| = \sum_{j=1}^m s_j,$$

$$\text{depth}(\vec{s}) := m,$$

$$\text{height}(\vec{s}) := \#\{j : s_j \geq 2\}.$$

### Theorem 14 (J. Okuda & Y. Takeyama)

$$\begin{aligned} 1 + (w-uv) \sum_{s,m,h \geq 0} u^{s-m-h} v^{m-h} w^{h-1} \sum_{\substack{\text{weight}(\vec{s})=s \\ \text{depth}(\vec{s})=m \\ \text{height}(\vec{s})=h}} \zeta[\vec{s}] \\ = \exp \left\{ \sum_{k=2}^{\infty} (u^k + v^k - \alpha^k - \beta^k) \frac{1}{k} \sum_{j=2}^k (q-1)^{k-j} \zeta[j] \right\}, \end{aligned}$$

where  $\alpha$  and  $\beta$  satisfy the equations

$$\alpha + \beta = u + v + (q-1)(w-uv), \quad \alpha\beta = w.$$

Theorem 13 is case  $w = 0$  of Theorem 14.

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$$\zeta(2, 1) = \sum_{n>m>0} n^{-2} m^{-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k+j)^{-2} k^{-1}$$

$$= \sum_{k=1}^{\infty} k^{-1} \sum_{j=1}^{\infty} (k+j)^{-1} \int_0^1 t^{k+j-1} dt$$

$$= \sum_{k=1}^{\infty} k^{-1} \int_0^1 t^{-1} \sum_{j=1}^{\infty} \int_0^t u^{k+j-1} du dt$$

$$= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} k^{-1} u^k du dt$$

$$= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} \int_0^u v^{k-1} dv du dt$$

$$= \int_{1>t>u>v>0} \frac{dt}{t} \cdot \frac{du}{1-u} \cdot \frac{dv}{1-v}$$

$$= \int_0^1 AB^2.$$

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## The Jackson Integral

Suppose  $f : (0, b] \rightarrow \mathbf{R}$  and  $0 < x \leq b$ .

The Jackson  $q$ -integral of  $f$  on the subinterval  $(0, x]$  is

$$\int_0^x f(t) d_q t := (1 - q) \sum_{j=0}^{\infty} x q^j f(x q^j).$$

If there exists  $0 \leq \alpha < 1$  such that  $|f(t)t^\alpha|$  is bounded on  $(0, b]$ , then the integral converges to a function  $F(x)$  on  $(0, b]$ .

Additionally,  $F$  is a  $q$ -antiderivative of  $f$ :

$$D_q F(x) := \frac{F(qx) - F(x)}{(q - 1)x} = f(x), \quad 0 < x \leq b.$$

Note that

$$\lim_{q \rightarrow 1} D_q F(x) = F'(x),$$

and

$$\lim_{q \rightarrow 1} \int_0^x f(t) d_q t = \int_0^x f(t) dt.$$

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## The Jackson Simplex Integral

Let  $s_1, \dots, s_m$  are positive integers. Recall:

$$\zeta(s_1, \dots, s_m) = \int \prod_{k=1}^m \left( \prod_{r=1}^{s_k-1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \dots > t_{s_1}^{(1)} > \dots > t_1^{(m)} > \dots > t_{s_m}^{(m)} > 0.$$

**Theorem 15**

$$\zeta[s_1, \dots, s_m] = \int \prod_{k=1}^m \left( \prod_{r=1}^{s_k-1} \frac{d_q t_r^{(k)}}{t_r^{(k)}} \right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}^{(k)}},$$

where

$$y_k := \prod_{j=1}^k q^{1-s_j},$$

and the integral is over the same simplex as above.

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## Duality

Let  $a_i, b_i \in \mathbf{Z}^+$  and  $k = \sum_{i=1}^n (a_i + b_i)$ . Then

$$\begin{aligned} & \zeta(a_1 + 1, \{1\}^{b_1-1}, \dots, a_n + 1, \{1\}^{b_n-1}) \\ &= \int_0^1 \prod_{i=1}^n A^{a_i} B^{b_i} \\ &= \int_{1 > t_1 > \dots > t_k > 0} \prod_{j=1}^k f_j(t_j) dt_j \\ &= \int_{1 > u_k > \dots > u_1 > 0} \prod_{j=1}^k f_j(u_j) du_j, \quad u_j = 1 - t_j \\ &= \int_0^1 \prod_{i=n}^1 A^{b_i} B^{a_i} \\ &= \zeta(b_n + 1, \{1\}^{a_n-1}, \dots, b_1 + 1, \{1\}^{a_1-1}). \end{aligned}$$

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## Generalized Duality

**Definition 16** Let  $n$  and  $s_1, \dots, s_n$  be positive integers with  $s_1 > 1$ . Let  $m$  be a non-negative integer. Define

$$S(s_1, \dots, s_n; m) := \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = m}} \zeta(s_1 + c_1, \dots, s_n + c_n).$$

For positive integers  $a_i$  and  $b_i$ , define the dual argument lists

$$\vec{s} = \text{Cat}_{i=1}^n \{a_i + 1, \{1\}^{b_i-1}\},$$

$$\vec{s}' = \text{Cat}_{i=n}^1 \{b_i + 1, \{1\}^{a_i-1}\}.$$

**Theorem 17 (Y. Ohno)** For any pair of dual argument lists  $\vec{s}, \vec{s}'$  and any non-negative integer  $m$ , we have the equality

$$S(\vec{s}; m) = S(\vec{s}'; m).$$

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## Generalized $q$ -Duality

**Definition 18** Let  $n$  and  $s_1, \dots, s_n$  be positive integers with  $s_1 > 1$ . Let  $m$  be a non-negative integer. Define

$$S[s_1, \dots, s_n; m] := \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = m}} \zeta[s_1 + c_1, \dots, s_n + c_n].$$

For positive integers  $a_i$  and  $b_i$ , define the dual argument lists

$$\begin{aligned} \vec{s} &= \mathbf{Cat}_{i=1}^n \{a_i + 1, \{1\}^{b_i-1}\} \\ \vec{s}' &= \mathbf{Cat}_{i=n}^1 \{b_i + 1, \{1\}^{a_i-1}\}. \end{aligned}$$

**Theorem 19** For any pair of dual argument lists  $\vec{s}, \vec{s}'$  and any non-negative integer  $m$ , we have

$$S[\vec{s}; m] = S[\vec{s}'; m].$$

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## $q$ -Duality

**Corollary 8** If  $\vec{s}, \vec{s}'$  are dual argument lists, then

$$\zeta[\vec{s}] = \zeta[\vec{s}'].$$

In other words, if  $a_i, b_i \in \mathbf{Z}^+$  ( $1 \leq i \leq n$ ), then

$$\begin{aligned} \zeta\left[\mathbf{Cat}_{i=1}^n \{a_i + 1, \{1\}^{b_i-1}\}\right] \\ = \zeta\left[\mathbf{Cat}_{i=n}^1 \{b_i + 1, \{1\}^{a_i-1}\}\right]. \end{aligned}$$

**Proof.** Put  $m = 0$  in Theorem 19 (generalized  $q$ -duality).  $\square$

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## $q$ -Sum Formula

**Definition 20** Let  $t_1, \dots, t_n$  be positive integers.

$$\zeta^*[t_1, \dots, t_n] := \zeta[t_1 + 1, \mathbf{Cat}_{j=2}^n t_j].$$

**Corollary 9 ( $q$ -Sum Formula)** For any integers  $0 < k \leq n$ , we have

$$\sum_{t_1 + t_2 + \dots + t_n = k} \zeta^*[t_1, t_2, \dots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers  $t_1, \dots, t_n$  with sum equal to  $k$ .

**Proof.** If we take the dual argument lists in the form  $\vec{s} = (n + 1)$  and  $\vec{s}' = (2, \{1\}^{n-1})$  and put  $m = k - n$ , then Theorem 19 states that

$$\begin{aligned} \zeta[k + 1] &= \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = k - n}} \zeta[2 + c_2, \mathbf{Cat}_{j=2}^n \{1 + c_j\}] \\ &= \sum_{\substack{t_1, \dots, t_n \geq 1 \\ t_1 + \dots + t_n = k}} \zeta[t_1 + 1, \mathbf{Cat}_{j=2}^n t_j]. \end{aligned}$$

$\square$

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## $q$ -Cyclic Sum Formula

**Definition 21** Let  $s_j \in \mathbf{Z}^+$  for  $1 \leq j \leq n$  and put  $\vec{s} = (s_1, \dots, s_n)$ . Let  $\sigma$  denote the  $n$ -cycle  $(1\ 2 \dots n)$ , and let

$$\mathcal{C}(\vec{s}) := \{(s_{\sigma^j(1)}, \dots, s_{\sigma^j(n)}) : 1 \leq j \leq n\}$$

denote the set of cyclic permutations of  $\vec{s}$ .

Recall the definition

$$\zeta^*[s_1, \dots, s_n] := \zeta[s_1 + 1, s_2, \dots, s_n].$$

**Theorem 22** Let  $\vec{s}$  and  $\vec{s}'$  be dual argument lists.

Then

$$\sum_{\vec{t} \in \mathcal{C}(\vec{s})} \zeta^*[\vec{t}] = \sum_{\vec{t} \in \mathcal{C}(\vec{s}')} \zeta^*[\vec{t}].$$

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## Proof of Generalized $q$ -Duality

Let  $\mathfrak{h} = \mathbf{Q}\langle x, y \rangle$  denote the non-commutative polynomial algebra over the rational numbers in two indeterminates  $x$  and  $y$ .

Let  $\mathfrak{h}^0$  denote the subalgebra  $\mathbf{Q}1 \oplus x\mathfrak{h}y$ .

The  $\mathbf{Q}$ -linear map  $\widehat{\zeta}$  is defined on  $\mathfrak{h}^0$  by

$$\widehat{\zeta}[1] := \zeta[1] = 1$$

and

$$\widehat{\zeta}\left[\prod_{i=1}^s x^{a_i} y^{b_i}\right] = \zeta\left[\mathbf{Cat}_{i=1}^s \{a_i + 1, \{1\}^{b_i-1}\}\right],$$

for positive integers  $a_i, b_i$  ( $1 \leq i \leq s$ ).

Let  $\tau$  be the anti-automorphism of  $\mathfrak{h}$  that switches  $x$  and  $y$ .

Then  $q$ -duality simply says that

$$\widehat{\zeta}[\tau w] = \widehat{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.$$

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For each  $n \in \mathbf{Z}^+$ , let  $D_n$  be the derivation on  $\mathfrak{h}$  that maps  $x \mapsto 0$  and  $y \mapsto x^n y$ .

Let  $\theta$  be an indeterminate (formal parameter).

Define

$$\Delta := \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n \quad \text{and} \quad \sigma := \exp(\Delta).$$

Then

$\Delta$  is a derivation on  $\mathfrak{h}[[\theta]]$ , and  $\sigma$  is an automorphism of  $\mathfrak{h}[[\theta]]$ .

For any word  $w \in \mathfrak{h}^0$ , define

$$f[w; \theta] := \widehat{\zeta}[\sigma w]$$

and

$$g[w; \theta] := f[\tau w; \theta] = \widehat{\zeta}[\sigma \tau w].$$

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Recall

$$D_n \text{ sends } x \mapsto 0, y \mapsto x^n y.$$

Thus,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n : x \mapsto 0, y \mapsto \{\log(1 - x\theta)^{-1}\}y$$

and

$$\sigma = \exp(\Delta) : x \mapsto x, y \mapsto (1 - x\theta)^{-1}y.$$

Therefore,

$$\begin{aligned} f\left[\prod_{i=1}^s x^{a_i} y^{b_i}; \theta\right] &= \widehat{\zeta}\left[\prod_{i=1}^s x^{a_i} \{(1 - x\theta)^{-1}y\}^{b_i}\right] \\ &= \sum_{m=0}^{\infty} \theta^m \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = m}} \zeta\left[\mathbf{Cat}_{j=1}^n \{t_j + c_j\}\right], \\ &= \sum_{m=0}^{\infty} \theta^m S[\vec{t}; m], \end{aligned}$$

where

$$\vec{t} = (t_1, \dots, t_n) = \left(\mathbf{Cat}_{i=1}^s \{a_i + 1, \{1\}^{b_i-1}\}\right), n = \sum_{i=1}^s b_i.$$

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## Generalized $q$ -Duality Reformulated

**Theorem 23** For all  $w \in \mathfrak{h}^0$ ,  $f[w; \theta] = g[w; \theta]$ , i.e.  $\widehat{\zeta} \circ \sigma$  is invariant under ordinary duality  $\tau$ .

**Theorem 24** Let  $a_i, b_i \in \mathbf{Z}^+$  and  $\sum_{i=1}^s (a_i + b_i) > 2$ . Let  $\theta' := q\theta - 1$ , and set

$$I^m = \underbrace{\{0, 1\} \times \dots \times \{0, 1\}}_m.$$

The generating functions  $f$  and  $g$  satisfy the difference equation

$$\begin{aligned} &\sum_{\substack{\epsilon, \delta \in I^s \\ \delta_1 < a_1, \epsilon_s < b_s}} (-\theta)^{\bar{\delta} \cdot \bar{\epsilon}} (1 - q)^{\delta \cdot \epsilon} f\left[\prod_{i=1}^s x^{a_i - \delta_i} y^{b_i - \epsilon_i}; \theta\right] \\ &= \sum_{\substack{\delta, \epsilon \in I^{s+1} \\ \delta_{s+1} = \epsilon_1 = 0 \\ \delta_1 < a_1, \epsilon_{s+1} < b_s}} (-\theta')^{\bar{\delta} \cdot \bar{\epsilon} - 1} (1 - q)^{\delta \cdot \epsilon} f\left[\prod_{i=1}^s x^{a_i - \delta_i} y^{b_i - \epsilon_i + 1}; \theta'\right]. \end{aligned}$$

Here,  $\bar{\delta}$  to denotes the ordered tuple whose  $i^{\text{th}}$  component is  $1 - \delta_i$ , and of course  $\delta \cdot \epsilon$  denotes the dot product  $\sum_i \delta_i \epsilon_i$ . Similarly,  $\bar{\epsilon}$  denotes the ordered tuple whose  $i^{\text{th}}$  component is  $1 - \epsilon_i$ , and  $\bar{\delta} \cdot \bar{\epsilon} = \sum_i (1 - \delta_i)(1 - \epsilon_i)$ .

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**Proof of Theorem 23.** Use induction on the total degree of the word  $\prod_{i=1}^s x^{a_i} y^{b_i}$ .

The base case is clearly satisfied, since the word  $xy$  is self-dual.

Now apply Theorem 24 to  $f$  and  $g$  and subtract the resulting two equations.

The terms whose words have total degree less than  $\sum_{i=1}^s (a_i + b_i)$  are cancelled by the induction hypothesis.

This leaves us with

$$\begin{aligned} & (-\theta)^s \left\{ f \left[ \prod_{i=1}^s x^{a_i} y^{b_i}; \theta \right] - g \left[ \prod_{i=1}^s x^{a_i} y^{b_i}; \theta \right] \right\} \\ &= (-\theta')^s \left\{ f \left[ \prod_{i=1}^s x^{a_i} y^{b_i}; \theta' \right] - g \left[ \prod_{i=1}^s x^{a_i} y^{b_i}; \theta' \right] \right\}. \end{aligned}$$

Thus, the function

$$H(\theta) := (-\theta)^s \left\{ f \left[ \prod_{i=1}^s x^{a_i} y^{b_i}; \theta \right] - g \left[ \prod_{i=1}^s x^{a_i} y^{b_i}; \theta \right] \right\}$$

satisfies the functional equation  $H(\theta) = H(\theta')$ , where  $\theta' = q\theta - 1$ .

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One can show that  $H(\theta)$  is a meromorphic function of  $\theta$  of the form

$$H(\theta) = \theta^s \sum_{n=1}^{\infty} \frac{h_n}{[n]_q - \theta q^n},$$

with at worst simple poles at  $\theta = p_n := q^{-n}[n]_q$  for positive integers  $n$ .

Note that

$$0 = p_0 < p_1 < p_2 < \cdots < p_{n-1} < p_n < \cdots$$

and

$$p_n' = qp_n - 1 = p_{n-1} \quad \text{for all } n \geq 1.$$

The functional equation

$$H(\theta) = H(\theta')$$

thus implies that if  $H$  has a pole at  $p_n$ , then  $H$  must also have a pole at  $p_{n-1}$ .

Since  $H$  has no pole at  $p_0$ , it follows that each  $h_n = 0$ .

Thus,  $H$  vanishes identically and so  $f = g$ .  $\square$

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## Derivations

**Definition 25 (K. Ihara & M. Kaneko)** Define a derivation on  $\mathfrak{h}$  for each positive integer  $n$  by

$$\partial_n(x) = x(x+y)^{n-1}, \quad \partial_n(y) = -x(x+y)^{n-1}y.$$

**Theorem 26 (Ihara & Kaneko)** For all positive integers  $n$  and words  $w \in \mathfrak{h}^0$ ,  $\widehat{\zeta}(\partial_n(w)) = 0$ .

**Theorem 27 ( $q$ -Analog)** For all positive integers  $n$  and words  $w \in \mathfrak{h}^0$ ,  $\widehat{\zeta}[\partial_n(w)] = 0$ .

Theorem 27 is actually *equivalent* to generalized  $q$ -duality (Theorem 23).

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## Proof of Theorem 27

**Proof.** Let  $\sigma = \exp(\Delta)$ ,  $\bar{\sigma} = \tau\sigma\tau$ ,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n, \quad \partial = \sum_{n=1}^{\infty} \frac{\partial_n}{n} \theta^n.$$

Generalized  $q$ -duality (Theorem 23):  $\forall w \in \mathfrak{h}^0$ ,

$$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w] = \widehat{\zeta}[\tau\sigma\tau w] \iff (\sigma - \bar{\sigma})w \in \ker \widehat{\zeta}.$$

We show that in fact,  $(\sigma - \bar{\sigma})\mathfrak{h}^0 = \partial\mathfrak{h}^0$ .

To prove this, we require the following identity of Ihara and Kaneko.

**Proposition 28**  $\exp(\partial) = \bar{\sigma}\sigma^{-1}$ .

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To complete the proof of Theorem 27, observe that since

$$\begin{aligned}\partial &= \log(\bar{\sigma}\sigma^{-1}) = \log(1 - (\sigma - \bar{\sigma})\sigma^{-1}) \\ &= -(\sigma - \bar{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n} ((\sigma - \bar{\sigma})\sigma^{-1})^{n-1} \sigma^{-1},\end{aligned}$$

and

$$\begin{aligned}\sigma - \bar{\sigma} &= (1 - \bar{\sigma}\sigma^{-1})\sigma = (1 - \exp(\partial))\sigma \\ &= -\partial \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{n!} \sigma,\end{aligned}$$

we see that

$$\partial\mathfrak{h}^0 \subseteq (\sigma - \bar{\sigma})\mathfrak{h}^0 \quad \text{and} \quad (\sigma - \bar{\sigma})\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0.$$

Thus for the kernel of  $\widehat{\zeta}$ , we have the equivalences

$$\begin{aligned}(\sigma - \bar{\sigma})w \in \ker \widehat{\zeta} &\iff \partial w \in \ker \widehat{\zeta} \\ &\iff \forall n \in \mathbf{Z}^+, \widehat{\zeta}[\partial_n w] = 0.\end{aligned}$$

□