On q-Analogs of Multiple Zeta Values and other Multiple Harmonic Series

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Multiple Harmonic Sums

$$Z_n^{\ge}(s_1,\dots,s_m) := \sum_{k_1=1}^{n}\frac{1}{k_1^{s_1}}\sum_{k_2=1}^{k_1}\frac{1}{k_2^{s_2}}\cdots\sum_{k_m=1}^{k_{m-1}}\frac{1}{k_m^{s_m}}$$

$$= \sum_{n\ge k_1\ge k_2\ge\cdots\ge k_m\ge 1}\prod_{j=1}^{m} k_j^{-s_j}$$

- $k_1 \ge k_2 \ge \cdots \ge k_m$ are positive integers
- n may be finite or infinite ($0 \le n \le \infty$)
- $s_1,\dots,s_m$ are positive integers ($\forall j, s_j \in \mathbf{Z}^+$)
- $s_1 > 1$ if $n = \infty$

$$Z_n^{>}(s_1,\dots,s_m) := \sum_{k_1=1}^{n}\frac{1}{k_1^{s_1}}\sum_{k_2=1}^{k_1-1}\frac{1}{k_2^{s_2}}\cdots\sum_{k_m=1}^{k_{m-1}-1}\frac{1}{k_m^{s_m}}$$

$$= \sum_{n\ge k_1> k_2>\cdots> k_m\ge 1}\prod_{j=1}^{m} k_j^{-s_j}$$

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- n may be finite or infinite ($0 \le n \le \infty$)
- $s_1,\dots,s_m$ are positive integers ($\forall j, s_j \in \mathbf{Z}^+$)
- $s_1 > 1$ if $n = \infty$
- $\zeta(s_1,\dots,s_m) := Z_\infty^{>}(s_1,\dots,s_m)$ is called a *multiple zeta value* (MZV)

Relationship between $Z_n^{\ge}$ and $Z_n^{>}$

$$Z_n^{\ge}(s) = Z_n^{>}(s),$$

$$Z_n^{\ge}(s,t) = Z_n^{>}(s,t) + Z_n^{>}(s+t),$$

$$Z_n^{\ge}(s,t,u) = Z_n^{>}(s,t,u) + Z_n^{>}(s+t,u)$$

$$+ Z_n^{>}(s,t+u) + Z_n^{>}(s+t+u).$$

More generally, let $\vec{s} = (s_1, s_2, \dots, s_m)$. Then

$$Z_n^{\ge}(\vec{s}) = \sum Z_n^{>}(\vec{t}),$$

where the sum is over all $\vec{t}$ obtained from $\vec{s}$ by replacing any number of commas by plus signs.

Positive q-Integers

Definition 1 *The q-analog of $k \in \mathbf{Z}^+$ is*

$$[k]_q := \sum_{j=0}^{k-1} q^j$$

$$= 1 + q + q^2 + \cdots + q^{k-1}$$

$$= \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

Note that $\lim_{q\to 1} [k]_q = k$.

Can we find reasonable/interesting q-analogs of multiple harmonic sums?

The q-Factorial

If we set

$$[n]!_q := \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q},$$

then evidently

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

and thus

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

We also have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{1-q^{n-k+j}}{1-q^j}, \quad 0 \le k \le n.$$

L. Van Hamme:

$$\sum_{k=1}^{n} \frac{q^k}{1-q^k} = \sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k+1)/2}}{1-q^k} \begin{bmatrix} n \\ k \end{bmatrix}.$$

K. Dilcher:

$$\sum_{k_1=1}^{n} \frac{q^{k_1}}{1-q^{k_1}} \sum_{k_2=1}^{k_1} \frac{q^{k_2}}{1-q^{k_2}} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m}}{1-q^{k_m}}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k+1)/2+(m-1)k}}{(1-q^k)^m} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Equivalently,

$$\sum_{n\ge k_1\ge\cdots\ge k_m\ge 1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k+1)/2+(m-1)k}}{[k]_q^m} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Multiple q-Harmonic Sums

Definition 2 *Let n, m and $s_1, s_2, \ldots, s_m$ be positive integers. Define*

$$Z_n[s_1, \ldots, s_m] := \sum_{n\ge k_1\ge\cdots\ge k_m\ge 1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}},$$

$$A_n[s_1, \ldots, s_m] := \sum_{n\ge k_1\ge\cdots\ge k_m\ge 1} (-1)^{k_1+1} \times q^{k_1(k_1+1)/2} \begin{bmatrix} n \\ k_1 \end{bmatrix} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Both sums vanish if $n = 0$. If $n > 0$ and $m = 0$, define $Z_n[\,] = A_n[\,] = 1$.

Abbreviations:

$\operatorname*{Cat}_{j=1}^{m}\{s_j\}$ denotes the sequence $s_1, s_2, \ldots, s_m$.

$$\{s\}^m := \operatorname*{Cat}_{j=1}^{m}\{s\} = \underbrace{s, \ldots, s}_{m}$$

(i.e. $m \ge 0$ consecutive copies of s).

Theorem 3 *Let* $n, r, a_1, b_1, \ldots, a_r, b_r \in \mathbf{Z}^+$. *Then*

$$Z_n\Big[\operatorname*{Cat}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\Big] = A_n\Big[a_1, \{1\}^{b_1-1}, \operatorname*{Cat}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\Big].$$

Example 1 *Putting* $r=2$, $a_1=3$, $b_1=2$, $a_2=b_2=1$ *gives* $Z_n[1,1,3,1] = A_n[3,1,2]$, *i.e.*

$$\sum_{n\ge j\ge k\ge m\ge p\ge 1} \frac{q^{j+k+m+p}}{[j]_q[k]_q[m]_q^3[p]_q} = \sum_{n\ge k\ge m\ge p\ge 1} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{2k+p}}{[k]_q^3[m]_q[p]_q^2}.$$

Example 2 *Putting* $r=2$, $a_1=a_2=b_1=1$, $b_2=2$ *in Theorem 3 gives* $Z_n[2,2] = A_n[1,2,1]$, *i.e.*

$$\sum_{n\ge k\ge m\ge 1} \frac{q^{k+m}}{[k]_q^2[m]_q^2} = \sum_{n\ge k\ge m\ge p\ge 1} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^m}{[k]_q[m]_q^2[p]_q}.$$

A_n/Z_n Duality I

Define an involution on the set $\mathcal{S}$ of finite sequences of positive integers as follows:

Let α be the map that sends a sequence in $\mathcal{S}$ to its sequence of partial sums.

Let β be the involution on strictly increasing sequences in $\mathcal{S}$ defined by

$$\beta(a_1, \ldots, a_k) = \{1, 2, 3, \ldots, a_k\} \setminus \{a_1, \ldots, a_{k-1}\}$$

arranged in increasing order.

Theorem 3 can now be restated as

$$Z_n[\vec{s}\,] = A_n[\alpha^{-1}\beta\alpha\vec{s}\,], \quad \forall \vec{s} \in \mathcal{S}, \quad 0 < n \le \infty.$$

A_n/Z_n Duality II

Let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ and $\mathfrak{h}' = \mathbf{Q}1 \oplus \mathfrak{h}y$ and fix $0 < n \le \infty$.

Define $\mathbf{Q}$-linear maps $\widehat{A}_n$ and $\widehat{Z}_n$ on $\mathfrak{h}'$ by

$$\widehat{A}_n[1] := A_n[\,] = 1,$$

$$\widehat{A}_n\left[\prod_{j=1}^{k} x^{s_j-1}y\right] := A_n[s_1, \ldots, s_k], \quad s_j \in \mathbf{Z}^+,$$

and similarly for $\widehat{Z}_n$.

Let J be the automorphism of $\mathfrak{h}$ that switches x and y.

Define an involution of $\mathfrak{h}'$ by

$$w^* = (Jw)x^{-1}y, \qquad \forall w \in \mathfrak{h}y.$$

Then Theorem 3 can be restated as

$$\widehat{A}_n[w] = \widehat{Z}_n[w^*], \qquad \forall w \in \mathfrak{h}'.$$

Recall Theorem 3:

$$Z_n\Big[\operatorname*{Cat}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\Big] = A_n\Big[a_1, \{1\}^{b_1-1}, \operatorname*{Cat}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\Big].$$

Corollary 1 *Let* $n, a, b \in \mathbf{Z}^+$. *Then*

$$Z_n\big[\{1\}^{a-1}, b\big] = A_n\big[a, \{1\}^{b-1}\big].$$

Proof. Put $r=1$ in Theorem 3. □

Example 3 *Putting* $b=1$ *and* $a=m$ *yields*

$$Z_n[\{1\}^m] = A_n[m],$$

which is Dilcher's result

$$\sum_{n\ge k_1\ge\cdots\ge k_m\ge 1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q} = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{(m-1)k}}{[k]_q^m}.$$

Recall Corollary 1: If $n, a, b \in \mathbf{Z}^+$ then

$$Z_n\big[\{1\}^{a-1}, b\big] = A_n\big[a, \{1\}^{b-1}\big].$$

Example 4 *Putting $a = 1$ and $b = m$ yields*

$$Z_n[m] = A_n[\{1\}^m],$$

i.e.

$$\sum_{k=1}^{n} \frac{q^k}{[k]_q^m}$$

$$= \sum_{n \ge k_1 \ge \cdots \ge k_m \ge 1} (-1)^{k_1+1} q^{k_1(k_1+1)/2} \begin{bmatrix} n \\ k_1 \end{bmatrix} \prod_{j=1}^{m} \frac{1}{[k_j]_q},$$

with limiting case

$$\sum_{k=1}^{n} \frac{1}{k^m} = \sum_{n \ge k_1 \ge \cdots \ge k_m \ge 1} (-1)^{k_1+1} \binom{n}{k_1} \prod_{j=1}^{m} \frac{1}{k_j}.$$

Proof of Theorem 3

By induction, it suffices to establish the base cases

$$A_n[\ ] = A_n[0] = 1 \text{ for } 0 < n \in \mathbf{Z}$$

and the following two recurrence relations:

Proposition 4 *Let n, m and $s_1, s_2, \dots, s_m \in \mathbf{Z}^+$. Then*

$$A_n[s_1, \dots, s_m] = \sum_{r=1}^{n} \frac{q^r}{[r]_q} A_r[s_1 - 1, s_2, \dots, s_m].$$

Proposition 5 *Let n, m and $s_2, s_3, \dots, s_m \in \mathbf{Z}^+$. Then*

$$A_n[0, s_2, s_3, \dots, s_m] = \frac{A_n[s_2 - 1, s_3, \dots, s_m]}{[n]_q}.$$

Multiple Zeta Values

$$\zeta(s_1, \dots, s_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}.$$

The multiple series is absolutely convergent if

$$\sum_{j=1}^{n} \Re(s_j) > n, \qquad n = 1, 2, \dots, m.$$

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Euler ($m = 2$):

$$2\zeta(s, 1) = s\zeta(s+1) - \sum_{j=1}^{s-2} \zeta(s-j)\zeta(j+1),$$

where $2 \le s \in \mathbf{Z}$.

Period One

For all non-negative integers n,

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\{4\}^n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!},$$

$$\zeta(\{6\}^n) = \frac{6 \cdot (2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(\{8\}^n) = \frac{8 \cdot (2\pi)^{8n}}{(8n+4)!} \times \left\{ \left(1 + \frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1 - \frac{1}{\sqrt{2}}\right)^{4n+2} \right\}.$$

More generally, let $k \in \mathbf{Z}^+$ and $\omega := e^{i\pi/k}$. Then

$$\sum_{n=0}^{\infty} (-1)^n x^{2kn} \zeta(\{2k\}^n) = \prod_{j=0}^{k-1} \frac{\sin(\pi x \omega^j)}{\pi x \omega^j}.$$

Period Two

For all non-negative integers n,

$$\zeta(\{3,1\}^n) = 4^{-n}\zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!},$$

$$\zeta(3,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n}\zeta(4k+3)\zeta(\{4\}^{n-k})$$
$$= \sum_{k=0}^{n}\frac{2\pi^{4k}}{(4k+2)!}\left(-\frac{1}{4}\right)^{n-k}\zeta(4n-4k+3),$$

$$\zeta(2,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n}(-1)^k\zeta(\{4\}^{n-k})\Big\{(4k+1)$$
$$\times\,\zeta(4k+2) - 4\sum_{j=1}^{k}\zeta(4j-1)\zeta(4k-4j+3)\Big\}.$$

Multiple q-Zeta Values

Definition 6 *Let m and $s_1, s_2, \ldots, s_m$ be positive integers and $0 < q < 1$. Define*

$$\zeta[s_1,\ldots,s_m] := \sum_{k_1>\cdots>k_m>0}\prod_{j=1}^{m}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Observe that

$$\lim_{q\to 1}\zeta[s_1,\ldots,s_m] = \zeta(s_1,\ldots,s_m),$$

where again,

$$\zeta(s_1,\ldots,s_m) = \sum_{k_1>\cdots>k_m>0}\prod_{j=1}^{m}\frac{1}{k_j^{s_j}}.$$

Also

$$\zeta[s]\zeta[t] = \zeta[s,t] + \zeta[t,s] + \zeta[s+t]$$
$$+ (1-q)\zeta[s+t-1].$$

Period-1 Sums Reduce

Theorem 7 *If n is a positive integer and $s > 1$, then*

$$n\zeta[\{s\}^n]$$
$$= \sum_{k=1}^{n}(-1)^{k+1}\zeta[\{s\}^{n-k}]\sum_{j=0}^{k-1}\binom{k-1}{j}(1-q)^j\zeta[ks-j].$$

Example 5 *With $n = 2$, we get*

$$2\zeta[s,s] = \zeta[s]\zeta[s] - \Big(\zeta[2s] + (1-q)\zeta[2s-1]\Big).$$

Corollary 2 *If n is a positive integer and $s > 1$, then*

$$n\zeta(\{s\}^n) = \sum_{k=1}^{n}(-1)^{k+1}\zeta(\{s\}^{n-k})\zeta(ks).$$

Let $\mathfrak{S}_n$ denote the group of $n!$ permutations of $\langle n\rangle = \{1,2,\ldots,n\}$.

Theorem 8 *Let n be a positive integer, and let $s_j > 1$ for $1 \le j \le n$. Then*

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\Big[\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\Big] = \sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{k=1}^{|\mathcal{P}|}(|P_k|-1)!$$
$$\times\sum_{\nu_k=0}^{|P_k|-1}\binom{|P_k|-1}{\nu_k}(1-q)^{\nu_k}\zeta[p_k-\nu_k],$$

where the outer sum on the right is over all unordered set partitions $\mathcal{P} = \{P_1,\ldots,P_m\}$ of $\langle n\rangle$, $1 \le m = |\mathcal{P}| \le n$, and $p_k = \sum_{j\in P_k}s_j$.

Corollary 3 (M. Hoffman)

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\Big(\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\Big)$$
$$= \sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{P\in\mathcal{P}}(|P|-1)!\,\zeta\Big(\sum_{j\in P}s_j\Big).$$

Parity Reduction

Theorem 9 *Let $m \in \mathbf{Z}^+$ and let $s_1, \dots, s_m$ be real numbers with $s_1 > 1$, $s_m > 1$, and $s_j \ge 1$ for $1 < j < m$. Then*

$$\zeta\Big[\mathop{\mathrm{Cat}}_{k=1}^{m} s_k\Big] + (-1)^m \zeta\Big[\mathop{\mathrm{Cat}}_{k=1}^{m} s_{m-k+1}\Big]$$

can be expressed as a $\mathbf{Z}[q]$-linear combination of multiple q-zeta values of depth less than m.

That is, the coefficients in the linear combination are polynomials in q with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

A Double Generating Function

Theorem 10

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} \zeta[m+2, \{1\}^n]$$
$$= 1 - \exp\left\{ \sum_{k=2}^{\infty} \left\{u^k + v^k - \Big(u + v + (1-q)uv\Big)^k\right\} \times \frac{1}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta[j] \right\}.$$

Corollary 4 *If $0 \le m, n \in \mathbf{Z}$, then*

$$\zeta[m+2, \{1\}^n] = \zeta[n+2, \{1\}^m].$$

Corollary 5 (q-Euler) *Let $0 \le m \in \mathbf{Z}$. Then*

$$2\zeta[m+2,1] = (m+2)\zeta[m+3] + (1-q)\, m\, \zeta[m+2] - \sum_{k=2}^{m+1} \zeta[m+3-k]\, \zeta[k].$$

$$m = 0 \implies \zeta[2,1] = \zeta[3].$$

The Jackson Integral

Suppose $f : (0, b] \to \mathbf{R}$ and $0 < x \le b$.

The Jackson q-integral of f is defined by

$$\int_0^x f(t)\, d_q t := (1-q) \sum_{j=0}^{\infty} x q^j f(xq^j).$$

If there exists $0 \le \alpha < 1$ such that $|f(t)t^{\alpha}|$ is bounded on $(0, b]$, then the integral converges to a function $F(x)$ on $(0, b]$.

Additionally, F is a q-antiderivative of f:

$$D_q F(x) := \frac{F(qx) - F(x)}{(q-1)x} = f(x), \quad 0 < x \le b.$$

Note that

$$\lim_{q \to 1} D_q F(x) = F'(x),$$

and

$$\lim_{q \to 1} \int_0^x f(t)\, d_q t = \int_0^x f(t)\, dt.$$

The Jackson Simplex Integral

Theorem 11 *Let $s_1, \dots, s_m$ are positive integers. Then*

$$\zeta[s_1, \dots, s_m] = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{d_q t_r^{(k)}}{t_r^{(k)}} \right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}},$$

where

$$y_k := \prod_{j=1}^{k} q^{1-s_j},$$

and the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0.$$

Generalized q-Duality

Definition 12 *Let n and $s_1, \ldots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S[s_1, \ldots, s_n; m] := \sum_{\substack{c_1,\ldots,c_n \ge 0 \\ c_1+\cdots+c_n=m}} \zeta[s_1+c_1, \ldots, s_n+c_n].$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \{a_i+1, \{1\}^{b_i-1}\}$$
$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \{b_i+1, \{1\}^{a_i-1}\}.$$

Theorem 13 *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have*

$$S[\vec{s}\,; m] = S[\vec{s}\,'; m].$$

q-Duality

Corollary 6 *If $\vec{s}$, $\vec{s}\,'$ are dual argument lists, then*

$$\zeta[\vec{s}\,] = \zeta[\vec{s}\,'].$$

In other words, if $a_i, b_i \in \mathbf{Z}^+$ $(1 \le i \le n)$, then

$$\zeta\Big[\operatorname*{Cat}_{i=1}^{n} \{a_i+1, \{1\}^{b_i-1}\}\Big] = \zeta\Big[\operatorname*{Cat}_{i=n}^{1} \{b_i+1, \{1\}^{a_i-1}\}\Big].$$

Proof. Put $m = 0$ in Theorem 13 (generalized q-duality). □

q-Sum Formula

Definition 14 *Let $t_1, \ldots, t_n$ be positive integers.*

$$\zeta^*[t_1, \ldots, t_n] := \zeta\Big[t_1+1, \operatorname*{Cat}_{j=2}^{n} t_j\Big].$$

Corollary 7 (q-Sum Formula) *For any integers $0 < k \le n$, we have*

$$\sum_{t_1+t_2+\cdots+t_n=k} \zeta^*[t_1, t_2, \ldots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers $t_1, \ldots, t_n$ with sum equal to k.

Proof. If we take the dual argument lists in the form $\vec{s} = (n+1)$ and $\vec{s}\,' = (2, \{1\}^{n-1})$ and put $m = k-n$, then Theorem 13 states that

$$\zeta[k+1] = \sum_{\substack{c_1,\ldots,c_n \ge 0 \\ c_1+\cdots+c_n=k-n}} \zeta\Big[2+c_2, \operatorname*{Cat}_{j=2}^{n} \{1+c_j\}\Big]$$
$$= \sum_{\substack{t_1,\ldots,t_n \ge 1 \\ t_1+\cdots+t_n=k}} \zeta\Big[t_1+1, \operatorname*{Cat}_{j=2}^{n} t_j\Big].$$

□

q-Cyclic Sum Formula

Definition 15 *Let $s_j \in \mathbf{Z}^+$ for $1 \le j \le n$ and put $\vec{s} = (s_1, \ldots, s_n)$. Let σ denote the n-cycle $(1\,2 \cdots n)$, and let*

$$\mathcal{C}(\vec{s}\,) := \{(s_{\sigma^j(1)}, \ldots, s_{\sigma^j(n)}) : 1 \le j \le n\}$$

denote the set of cyclic permutations of $\vec{s}$.

Recall the definition

$$\zeta^*[s_1, \ldots, s_n] := \zeta[s_1+1, s_2, \ldots, s_n].$$

Theorem 16 *Let $\vec{s}$ and $\vec{s}\,'$ be dual argument lists. Then*

$$\sum_{\vec{t} \in \mathcal{C}(\vec{s}\,)} \zeta^*[\vec{t}\,] = \sum_{\vec{t} \in \mathcal{C}(\vec{s}\,')} \zeta^*[\vec{t}\,].$$

Proof of Generalized q-Duality

Let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y.

Let $\mathfrak{h}^0$ denote the subalgebra $\mathbf{Q}1 \oplus x\mathfrak{h}y$.

The $\mathbf{Q}$-linear map $\widehat{\zeta}$ is defined on $\mathfrak{h}^0$ by

$$\widehat{\zeta}[1] := \zeta[\,] = 1$$

and

$$\widehat{\zeta}\left[\prod_{i=1}^{s} x^{a_i}y^{b_i}\right] = \zeta\left[\operatorname*{Cat}_{i=1}^{s}\left\{a_i+1, \{1\}^{b_i-1}\right\}\right],$$

for positive integers a_i, b_i $(1 \le i \le s)$.

Let τ be the anti-automorphism of $\mathfrak{h}$ that switches x and y.

Then q-duality simply says that

$$\widehat{\zeta}[\tau w] = \widehat{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.$$

For each $n \in \mathbf{Z}^+$, let D_n be the derivation on $\mathfrak{h}$ that maps $x \mapsto 0$ and $y \mapsto x^n y$.

Let θ be an indeterminate (formal parameter).

Define

$$\Delta := \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n \quad \text{and} \quad \sigma := \exp(\Delta).$$

Then

Δ is a derivation on $\mathfrak{h}[[\theta]]$, and
σ is an automorphism of $\mathfrak{h}[[\theta]]$.

Theorem 13 (generalized q-duality) can now be reformulated as

$$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w], \qquad \forall w \in \mathfrak{h}^0.$$

In other words, $\widehat{\zeta} \circ \sigma$ is invariant under ordinary duality τ.

Derivations

Definition 17 (K. Ihara & M. Kaneko) *Define a derivation on $\mathfrak{h}$ for each positive integer n by*

$$\partial_n(x) = x(x+y)^{n-1}, \qquad \partial_n(y) = -x(x+y)^{n-1}y.$$

Theorem 18 (Ihara & Kaneko) *For all positive integers n and words $w \in \mathfrak{h}^0$*

$$\widehat{\zeta}(\partial_n(w)) = 0.$$

Theorem 19 (q-Analog) *For all positive integers n and words $w \in \mathfrak{h}^0$,*

$$\widehat{\zeta}[\partial_n(w)] = 0.$$

Theorem 19 is actually *equivalent* to generalized q-duality (Theorem 13).

Proof of Theorem 19

Proof. Let $\sigma = \exp(\Delta)$, $\tilde{\sigma} = \tau\sigma\tau$,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n, \qquad \partial = \sum_{n=1}^{\infty} \frac{\partial_n}{n}\theta^n.$$

Generalized q-duality (Theorem 13): $\forall w \in \mathfrak{h}^0$,

$$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w] = \widehat{\zeta}[\tau\sigma\tau w] \iff (\sigma - \tilde{\sigma})w \in \ker \widehat{\zeta}.$$

We show that in fact, $(\sigma - \tilde{\sigma})\mathfrak{h}^0 = \partial\mathfrak{h}^0$.

To prove this, we require the following identity of Ihara and Kaneko.

Proposition 20 $\exp(\partial) = \tilde{\sigma}\sigma^{-1}$.

To complete the proof of Theorem 19, observe that since

$$\partial = \log\left(\tilde{\sigma}\sigma^{-1}\right) = \log\left(1 - (\sigma - \tilde{\sigma})\sigma^{-1}\right)$$

$$= -(\sigma - \tilde{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n}\Big((\sigma - \tilde{\sigma})\sigma^{-1}\Big)^{n-1}\sigma^{-1},$$

and

$$\sigma - \tilde{\sigma} = \left(1 - \tilde{\sigma}\sigma^{-1}\right)\sigma = \left(1 - \exp(\partial)\right)\sigma$$
$$= -\partial \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{n!}\sigma,$$

we see that

$$\partial\mathfrak{h}^0 \subseteq (\sigma - \tilde{\sigma})\mathfrak{h}^0 \quad \text{and} \quad (\sigma - \tilde{\sigma})\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0.$$

Thus for the kernel of $\widehat{\zeta}$, we have the equivalences

$$(\sigma - \tilde{\sigma})w \in \ker\widehat{\zeta} \quad \Longleftrightarrow \quad \partial w \in \ker\widehat{\zeta}$$
$$\Longleftrightarrow \quad \forall n \in \mathbf{Z}^+,\ \widehat{\zeta}[\partial_n w] = 0.$$

□