q-Analogues

David M. Bradley
University of Maine

http://www.math.umaine.edu/faculty/bradley/

October 26, 2006

Partitions

Let n be a positive integer.

Definition. A *partition* of n is an ordered tuple of positive integers whose components are arranged in *increasing* order and which sum to n.

Example. The partitions of 4 are (4), (1, 3), (2, 2), (1, 1, 2) and (1, 1, 1, 1).

- Denote the number of partitions of n by $p(n)$.
- Since there are 5 partitions of 4, $p(4) = 5$.

L. Euler (1748): Let $|q| < 1$. Then

$$1 + \sum_{n=1}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} (1 - q^k)^{-1}.$$

Proof. Expand each factor

$$(1 - q^k)^{-1} = 1 + q^k + q^{2k} + q^{3k} + \cdots = \sum_{\alpha=0}^{\infty} q^{\alpha k}$$

as a geometric series with ratio q^k:

$$\begin{aligned}
(1-q)^{-1} &= 1 + q^1 + q^{2\cdot 1} + q^{3\cdot 1} + q^{4\cdot 1} + \cdots \\
(1-q^2)^{-1} &= 1 + q^2 + q^{2\cdot 2} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots \\
(1-q^3)^{-1} &= 1 + q^3 + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots \\
(1-q^4)^{-1} &= 1 + \boxed{q^4} + q^{2\cdot 4} + q^{3\cdot 4} + q^{4\cdot 4} + \cdots \\
&\cdots \\
(1-q^k)^{-1} &= 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + q^{4\cdot k} + \cdots \\
&\cdots
\end{aligned}$$

The boxed term contributes the partition (4) to the coefficient of q^4.

Proof. Expand each factor

$$(1 - q^k)^{-1} = 1 + q^k + q^{2k} + q^{3k} + \cdots = \sum_{\alpha=0}^{\infty} q^{\alpha k}$$

as a geometric series with ratio q^k:

$$\begin{aligned}
(1-q^1)^{-1} &= 1 + \boxed{q^1} + q^{2\cdot 1} + q^{3\cdot 1} + q^{4\cdot 1} + \cdots \\
(1-q^2)^{-1} &= 1 + q^2 + q^{2\cdot 2} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots \\
(1-q^3)^{-1} &= 1 + \boxed{q^3} + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots \\
(1-q^4)^{-1} &= 1 + q^4 + q^{2\cdot 4} + q^{3\cdot 4} + q^{4\cdot 4} + \cdots \\
&\cdots \\
(1-q^k)^{-1} &= 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + q^{4\cdot k} + \cdots \\
&\cdots
\end{aligned}$$

The boxed terms contribute the partition (1, 3) to the coefficient of q^4.

Proof. Expand each factor

$$(1-q^k)^{-1} = 1 + q^k + q^{2k} + q^{3k} + \cdots = \sum_{\alpha=0}^{\infty} q^{\alpha k}$$

as a geometric series with ratio q^k:

$$\begin{aligned}
(1-q^1)^{-1} &= 1 + q^1 + q^{2\cdot 1} + q^{3\cdot 1} + q^{4\cdot 1} + \cdots \\
(1-q^2)^{-1} &= 1 + q^2 + \boxed{q^{2\cdot 2}} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots \\
(1-q^3)^{-1} &= 1 + q^3 + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots \\
(1-q^4)^{-1} &= 1 + q^4 + q^{2\cdot 4} + q^{3\cdot 4} + q^{4\cdot 4} + \cdots \\
&\cdots \\
(1-q^k)^{-1} &= 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + q^{4\cdot k} + \cdots \\
&\cdots
\end{aligned}$$

The boxed term contributes the partition $(2,2)$ to the coefficient of q^4.

Proof. Expand each factor

$$(1-q^k)^{-1} = 1 + q^k + q^{2k} + q^{3k} + \cdots = \sum_{\alpha=0}^{\infty} q^{\alpha k}$$

as a geometric series with ratio q^k:

$$\begin{aligned}
(1-q^1)^{-1} &= 1 + q^1 + \boxed{q^{2\cdot 1}} + q^{3\cdot 1} + q^{4\cdot 1} + \cdots \\
(1-q^2)^{-1} &= 1 + \boxed{q^2} + q^{2\cdot 2} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots \\
(1-q^3)^{-1} &= 1 + q^3 + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots \\
(1-q^4)^{-1} &= 1 + q^4 + q^{2\cdot 4} + q^{3\cdot 4} + q^{4\cdot 4} + \cdots \\
&\cdots \\
(1-q^k)^{-1} &= 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + q^{4\cdot k} + \cdots \\
&\cdots
\end{aligned}$$

The boxed terms contribute the partition $(1,1,2)$ to the coefficient of q^4.

Proof. Expand each factor

$$(1-q^k)^{-1} = 1 + q^k + q^{2k} + q^{3k} + \cdots = \sum_{\alpha=0}^{\infty} q^{\alpha k}$$

as a geometric series with ratio q^k:

$$\begin{aligned}
(1-q^1)^{-1} &= 1 + q^1 + q^{2\cdot 1} + q^{3\cdot 1} + \boxed{q^{4\cdot 1}} + \cdots \\
(1-q^2)^{-1} &= 1 + q^2 + q^{2\cdot 2} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots \\
(1-q^3)^{-1} &= 1 + q^3 + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots \\
(1-q^4)^{-1} &= 1 + q^4 + q^{2\cdot 4} + q^{3\cdot 4} + q^{4\cdot 4} + \cdots \\
&\cdots \\
(1-q^k)^{-1} &= 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + q^{4\cdot k} + \cdots \\
&\cdots
\end{aligned}$$

The boxed term contributes the partition $(1,1,1,1)$ to the coefficient of q^4.

Proof. Expand each factor

$$(1-q^k)^{-1} = 1 + q^k + q^{2k} + q^{3k} + \cdots = \sum_{\alpha_k=0}^{\infty} q^{\alpha_k \cdot k}$$

as a geometric series with ratio q^k:

$$\begin{aligned}
(1-q^1)^{-1} &= 1 + q^1 + q^{2\cdot 1} + q^{3\cdot 1} + q^{4\cdot 1} + \cdots \\
(1-q^2)^{-1} &= 1 + q^2 + q^{2\cdot 2} + q^{3\cdot 2} + q^{4\cdot 2} + \cdots \\
(1-q^3)^{-1} &= 1 + q^3 + q^{2\cdot 3} + q^{3\cdot 3} + q^{4\cdot 3} + \cdots \\
&\cdots \\
(1-q^k)^{-1} &= 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + q^{4\cdot k} + \cdots \\
&\cdots
\end{aligned}$$

In general, the contribution to the coefficient of q^n is the sum of all terms of the form

$$q^{\alpha_1 \cdot 1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 + \cdots + \alpha_k \cdot k + \cdots}.$$

Each α_k is a non-negative integer, and

$$\sum_{k=1}^{\infty} \alpha_k \cdot k = n.$$

Each term (*) corresponds to a partition of n in which the number of parts equal to k is α_k. $\square$

Partitions with Distinct Parts

Notation. Let $p_d(n)$ denote the number of partitions of the positive integer n in which all parts are *distinct*.

Example. There are only 2 partitions of 4 with all parts distinct: (4) and (1, 3). Thus $p_d(4) = 2$.

Theorem. Let $|q| < 1$. Then

$$1 + \sum_{n=1}^{\infty} p_d(n) q^n = \prod_{k=1}^{\infty} (1 + q^k).$$

Partitions with Distinct Parts

Notation. Let $p_d(n)$ denote the number of partitions of the positive integer n in which all parts are *distinct*.

Theorem. Let $|q| < 1$. Then

$$1 + \sum_{n=1}^{\infty} p_d(n) q^n = \prod_{k=1}^{\infty} (1 + q^k).$$

Proof.

$$\prod_{k=1}^{\infty} (1+q^k) = (1+q)(1+q^2)(1+q^3)(1+q^4)+\cdots$$

The contribution to the coefficient of q^n consists of all terms

$$q^{k_1+k_2+k_3+\cdots}$$

with $k_1 < k_2 < k_3 < \cdots$ and $k_1 + k_2 + k_3 + \cdots = n$.

The set of all such terms is in one-to-one correspondence with the set of all tuples $(k_1, k_2, k_3, \ldots)$ each of which is a partition of n with distinct parts.

□

Pentagonal Number Theorem

Euler (1750): Let $|q| < 1$. Then

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \prod_{k=1}^{\infty} (1 - q^k).$$

The numbers $n(3n+1)/2$, where n is an integer, are called the *pentagonal numbers*.

The pentagonal number series is a special case of **Jacobi's** theta series

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2},$$

in which q has been replaced by $q^{3/2}$ and z by $-q^{1/2}$.

Pentagonal Number Theorem

Euler (1750): Let $|q| < 1$. Then

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \prod_{k=1}^{\infty} (1 - q^k).$$

A. M. Legendre found the following *combinatorial* interpretation of Euler's result:

Let $p_e(m)$ denote the number of partitions of m into an *even* number of *distinct* parts.

Let $p_o(m)$ denote the number of partitions of m into an *odd* number of *distinct* parts.

Then

$$p_e(m) - p_o(m) = \begin{cases} (-1)^n, & \text{if } m = n(3n+1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

Recursion for $p(m)$

Recall that if $|q| < 1$, then

$$1 + \sum_{n=1}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} (1 - q^k)^{-1},$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \prod_{k=1}^{\infty} (1 - q^k).$$

It follows that

$$\left\{1 + \sum_{n=1}^{\infty} p(n)q^n\right\} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = 1.$$

Comparing coefficients of q^m for $m > 0$ gives

$$p(m) + \sum_{|n|=1}^{\infty} (-1)^n p(m - n(3n+1)/2) = 0,$$

i.e.

$$p(m) = p(m-1) + p(m-2) - p(m-5) - p(m-7) + \cdots$$

Square and Triangular Numbers

C. F. Gauss: Let $|q| < 1$. Then

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{k=1}^{\infty} (1 - q^{2k})(1 + q^{2k-1})^2,$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{k=1}^{\infty} (1 - q^{2k})(1 - q^{2k-1})^2,$$

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{k=1}^{\infty} (1 - q^{2k})(1 - q^{2k-1})^{-1}.$$

All three results are special cases of a celebrated two-parameter identity of **C. G. Jacobi**:

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = \prod_{k=1}^{\infty} (1 - q^k)(1 + zq^k)(1 + z^{-1}q^{k-1}).$$

The first two Gauss identities follow from Jacobi's by replacing q by q^2 and z by $\pm 1/q$.

The third can be obtained from Jacobi's identity by substituting $z = 1$.

Recall Jacobi's Triple Product Identity:

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = \prod_{k=1}^{\infty} (1 - q^k)(1 + zq^k)(1 + z^{-1}q^{k-1}).$$

Setting $z = 1$ gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} &= \prod_{k=1}^{\infty} \boxed{(1 - q^k)(1 + q^k)}(1 + q^{k-1}) \\ &= 2\prod_{k=1}^{\infty} \boxed{(1 - q^{2k})}(1 + q^k) \\ &= 2\prod_{k=1}^{\infty} (1 - q^{2k})\left(\frac{1 - q^{2k}}{1 - q^k}\right) \\ &= 2\prod_{k=1}^{\infty} \frac{(1 - q^{2k})^2}{(1 - q^{2k})(1 - q^{2k-1})} \\ &= 2\prod_{k=1}^{\infty} \frac{1 - q^{2k}}{1 - q^{2k-1}}.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{k=1}^{\infty} \frac{1 - q^{2k}}{1 - q^{2k-1}}.$$

Jacobi Implies Euler

Let $|q| < 1$ and let z be any complex number. Then

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = \prod_{k=1}^{\infty} (1 - q^k)(1 + zq^k)(1 + z^{-1}q^{k-1}).$$

Since

$$q^{-n} q^{3n(n+1)/2} = q^{n(3n+1)/2},$$

Euler's pentagonal number theorem can be obtained by replacing q by q^3 and z by $-1/q$:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} &= \prod_{k=1}^{\infty} (1 - q^{3k})(1 - q^{3k-1})(1 - q^{3k-2}) \\ &= \prod_{k=1}^{\infty} (1 - q^k).\end{aligned}$$

The 2- and 4-Square Theorems

In a 1750 letter to Goldbach, Euler suggested that the most natural way to prove that every positive integer is a sum of 4 squares would be to expand

$$\left(\sum_{n=0}^{\infty} q^{n^2}\right)^4 = \left(1+q^1+q^4+q^9+\cdots\right)^4 = \sum_{m=0}^{\infty} c_m q^m$$

and show that $c_m > 0$ for each m.

Jacobi (1828): Let $|q|<1$. Then

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1+4\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1-q^{2n+1}},$$

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1+8\sum_{\substack{n=1\\ 4\nmid n}}^{\infty} \frac{nq^n}{1-q^n}.$$

Jacobi (1828): Let $|q|<1$. Then

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1+4\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1-q^{2n+1}}.$$

Denote the coefficient of q^m on the LHS by $r_2(m)$.

Then $r_2(m)$ denotes the number of representations of m as a sum of large 2 *integer* squares. That is,

$$r_2(m) = \#\{(n_1,n_2)\in \mathbf{Z}\times\mathbf{Z} : n_1^2+n_2^2=m\}.$$

But the series on the right is equal to

$$1+4\sum_{n=0}^{\infty}(-1)^n \sum_{k=1}^{\infty} q^{(2n+1)k}.$$

If n is even, then $2n+1\equiv 1 \pmod 4$, while if n is odd, then $2n+1\equiv 3 \pmod 4$. Therefore, for $m>0$,

$$r_2(m) = 4d_1(m) - 4d_3(m),$$

where $d_j(n)$ denotes the number of positive integer divisors of m congruent to j modulo 4.

Jacobi (1828): Let $|q|<1$. Then

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1+8\sum_{\substack{n=1\\ 4\nmid n}}^{\infty} \frac{nq^n}{1-q^n}.$$

Denote the coefficient of q^m on the LHS by $r_4(m)$, the number of representations of m as a sum of 4 *integer* squares.

That is,

$$r_4(m) = \#\{(n_1,n_2,n_3,n_4)\in \mathbf{Z}^4 : \sum_{j=1}^{4} n_j^2 = m\}.$$

The series on the right is equal to

$$1+8\sum_{\substack{n=1\\ n\neq 0 \bmod 4}}^{\infty} n \sum_{k=1}^{\infty} q^{nk}.$$

Therefore, for $m>0$,

$$r_4(m) = 8\sum_{\substack{d|m\\ d\neq 0 \bmod 4}} d.$$

Rogers-Ramanujan Identities

L. J. Rogers (1894): Let $|q|<1$. Then

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty}\frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty}\frac{1}{(1-q^{5m+2})(1-q^{5m+3})}.$$

Rediscovered by Ramanujan (1913), I. Schur (1917) and Rodney Baxter (1979).

Percy MacMahon (c. 1915): The number of partitions of n with minimal difference 2 is equal to the number of partitions of n with parts congruent to 1 or 4 modulo 5.

Ramanujan expressed the ratio of the two Rogers series as a continued fraction:

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\ddots}}} = \frac{\sum_{n=0}^{\infty} q^{n^2}\Big/(q;q)_n}{\sum_{n=0}^{\infty} q^{n(n+1)/2}\Big/(q;q)_n}.$$

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\ddots}}} = \left\{\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}\right\} e^{2\pi/5},$$

Let $\tau = (1+\sqrt{5})/2$ and $q = e^{-2\pi\sqrt{5}}$. Then

$$\cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\ddots}}} = \left(\frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\tau^{-5/2}-1}} - \tau\right).$$

G. H. Hardy: "A single look is enough to show that they could only be written down by a mathematician of the highest class."

Hypergeometric Functions

A generalized hypergeometric function has a series representation

$$\sum_{n=0}^{\infty} c_n, \qquad \frac{c_{n+1}}{c_n} = \text{rational function of } n.$$

Factor the ratio as

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)\,x}{(n+b_1)(n+b_2)\cdots(n+b_q)(n+1)}.$$

Define the shifted factorial

$$(a)_0 := 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n \geq 1.$$

If $c_0 = 1$, then c_n has the explicit representation

$$c_n = \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\cdot\frac{x^n}{n!},$$

and

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{x^n}{n!}.$$

Pfaff (1797): Let n be a non-negative integer.

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ c, a+b+1-c-n \end{matrix}\,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Definition. (Gamma Function): For complex $z \neq 0, -1, -2, -3, \ldots,$

$$\Gamma(z) := \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{n!\, n^z}{(z)_{n+1}}.$$

Gauss (1813): Let $\Re(c-a-b) > 0$. Then

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Proof. Let $n \to \infty$ in Pfaff's formula. Replacing n by $n+1$ on the right gives

$$\frac{\boxed{n!\, n^c}}{(c)_{n+1}} \cdot \frac{\boxed{n!\, n^{c-a-b}}}{(c-a-b)_{n+1}} \cdot \frac{(c-a)_{n+1}}{\boxed{n!\, n^{c-a}}} \cdot \frac{(c-b)_{n+1}}{\boxed{n!\, n^{c-b}}}.$$

$\square$

Pfaff's formula was rediscovered by Saalchütz in 1890.

Pfaff (1797): Let n be a non-negative integer.

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ c, a+b+1-c-n \end{matrix}\,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Definition. (Gamma Function): For complex $z \neq 0, -1, -2, -3, \ldots,$

$$\Gamma(z) := \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{n!\, n^z}{(z)_{n+1}}.$$

Gauss (1813): Let $\Re(c-a-b) > 0$. Then

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Proof. Let $n \to \infty$ in Pfaff's formula. Replacing n by $n+1$ on the right gives

$$\boxed{\frac{n!\, n^c}{(c)_{n+1}}} \cdot \boxed{\frac{n!\, n^{c-a-b}}{(c-a-b)_{n+1}}} \cdot \boxed{\frac{(c-a)_{n+1}}{n!\, n^{c-a}}} \cdot \boxed{\frac{(c-b)_{n+1}}{n!\, n^{c-b}}}.$$

$\square$

Pfaff's formula was rediscovered by Saalchütz in 1890.

Clausen (1828):

$$\left\{ {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+1/2 \end{matrix} \middle| x \right) \right\}^2 = {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b \\ a+b+1/2,\ 2a+2b \end{matrix} \middle| x \right).$$

Bieberbach (1916): If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is holomorphic and one-to-one in the unit disc, is it true that for each n, $|a_n| \le n$?

Louis de Branges (1985): Yes.

The final step in de Branges' proof required knowing that the integral of a certain Jacobi polynomial is non-negative.

The polynomial in question turned out to be a terminating ${}_3F_2$ to which Clausen's formula applies.

Ramanujan:

$$\sum_{n=0}^{\infty} \frac{(1103 + 26390n)(1/4)_n (1/2)_n (3/4)_n}{(n!)^3 \, (99)^{4n}} = \frac{9801}{2\pi\sqrt{2}}.$$

Familiar Hypergeometric Functions

For any complex number z,

$${}_0F_0\left(\middle| z \right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

If $|z| < 1$, then

$${}_1F_0\left(\begin{matrix} 1 \\ \ \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} (1)_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} z^n = (1-z)^{-1},$$

$${}_2F_1\left(\begin{matrix} a,\ b \\ b \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \binom{a+n-1}{n} z^n = (1-z)^{-a}.$$

If $0 < |z| < 1$, then

$${}_2F_1\left(\begin{matrix} 1,\ 1 \\ 2 \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{n!\,n!}{(n+1)!} \cdot \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n+1}$$

$$= z^{-1} \log(1-z)^{-1}.$$

Familiar Hypergeometric Functions

If $0 < |z| < 1$, then

$${}_2F_1\left(\begin{matrix} 1/2,\ 1 \\ 3/2 \end{matrix} \middle| -z^2 \right) = \sum_{n=0}^{\infty} \frac{(1/2)_n (1)_n}{(3/2)_n} \cdot \frac{(-z^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n+1}$$

$$= \frac{\arctan z}{z}.$$

For any complex z,

$${}_0F_1\left(\begin{matrix} \ \\ 1/2 \end{matrix} \middle| -\frac{z^2}{4} \right) = \cos z$$

$${}_0F_1\left(\begin{matrix} \ \\ 3/2 \end{matrix} \middle| -\frac{z^2}{4} \right) = \frac{\sin z}{z}.$$

Basic Hypergeometric Series

E. Heine (1846): Let $|z| < 1$, $|q| < 1$ and $c \neq 0, -1, -2, \ldots$. Consider the series

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)} z^2 + \cdots.$$

Since

$$\lim_{q \to 1} \frac{1-q^a}{1-q} = a,$$

Heine's series tends termwise to the Gauss hypergeometric series

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| z \right) = 1 + \frac{ab}{c \cdot 1!} z + \frac{a(a+1)b(b+1)}{c(c+1) \cdot 2!} z^2 + \cdots.$$

Basic Hypergeometric Series

E. Heine (1846): Let $|z| < 1$, $|q| < 1$ and $c \neq 0, -1, -2, \ldots$. Consider the series

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)} z$$

$$+ \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)} z^2 + \cdots .$$

Since we'd like to be able to replace q^a, q^b, q^c by zero, it is now customary to define the *basic hypergeometric series* by

$${}_2\phi_1\left(\begin{matrix} a,\, b \\ c,\, q \end{matrix}\,\middle|\, z\right) = 1 + \frac{(1-a)(1-b)}{(1-c)(1-q)} z$$

$$+ \frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-c)(1-cq)(1-q)(1-q^2)} z^2 + \cdots ,$$

where now $c \neq q^{-m}$ for any non-negative integer m.

Basic Hypergeometric Series

Recall the basic ${}_2\phi_1$ for $c \neq q^{-m}$, $m = 0, 1, 2, \ldots$:

$${}_2\phi_1\left(\begin{matrix} a,\, b \\ c,\, q \end{matrix}\,\middle|\, z\right) = 1 + \frac{(1-a)(1-b)}{(1-q)(1-c)} z$$

$$+ \frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)} z^2 + \cdots .$$

Introduce the q-shifted factorial

$$(a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}), \quad 0 \leq n \leq \infty.$$

Then

$${}_2\phi_1\left(\begin{matrix} a,\, b \\ c,\, q \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(q;q)_n} z^n.$$

q-Binomial Theorem

E. Heine (1847): For $|q| < 1$, $|b| < 1$, $|z| < 1$,

$${}_2\phi_1\left(\begin{matrix} a,\, b \\ c,\, q \end{matrix}\,\middle|\, z\right) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/b,\, z \\ az,\, q \end{matrix}\,\middle|\, b\right).$$

Taking $b = c$, we recover

H. Rothe (1811): "q-binomial theorem"

$${}_1\phi_0\left(\begin{matrix} a \\ q \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

The q-Binomial Theorem

H. Rothe (1811): "q-binomial theorem"

$${}_1\phi_0\left(\begin{matrix} a \\ q \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Letting $a = q^\alpha$ and $q \to 1$ yields the ordinary binomial theorem in the form

$$\sum_{n=0}^{\infty} \frac{(1-q^\alpha)(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} z^n$$

$$\to \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} z^n = (1-z)^{-\alpha}.$$

q-Calculus

Let $f : \mathbf{R} \to \mathbf{R}$ and let $q \neq 1$.

Definition. The q-*difference* operator d_q is defined by

$$(d_q f)(x) = f(qx) - f(x).$$

Example. For the identity map ι, since $\iota(x) = x$, we have

$$(d_q \iota)(x) = qx - x = (q-1)x.$$

Definition. The q-*derivative* operator D_q is defined by

$$(D_q f)(x) = \frac{(d_q f)(x)}{(d_q \iota)(x)} = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0.$$

Notation. If $y = f(x)$, then

$$\frac{d_q\, y}{d_q\, x} = (D_q f)(x).$$

The q-Derivative

Recall that if $y = f(x)$, then the q-derivative is

$$\frac{d_q\, y}{d_q\, x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

If f is differentiable at $x \neq 0$, then we recover the ordinary derivative $f'(x)$ from the q-derivative in the limit as $q \to 1$.

To see this, substitute $q = 1 + h/x$. Then

$$\lim_{q \to 1} \frac{f(qx) - f(x)}{(q-1)x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The q-Product Rule

Recall that if $y = f(x)$, then the q-derivative is

$$\frac{d_q\, y}{d_q\, x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Therefore, the q-derivative of the product of two functions f, g is given by

$$\begin{aligned}
&(D_q fg)(x) \\
&= \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x} \\
&= \frac{f(qx)[g(qx) - g(x)] + [f(qx) - f(x)]g(x)}{(q-1)x} \\
&= f(qx)(D_q g)(x) + g(x)(D_q f)(x).
\end{aligned}$$

The q-Integral

J. Thomae (1869): Let $0 < q < 1$. Then

$$\int_0^1 f(t)\, d_q t := (1-q) \sum_{n=0}^{\infty} q^n f(q^n).$$

F. H. Jackson (1910):

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,$$

$$\int_0^b f(t)\, d_q t := (1-q) \sum_{n=0}^{\infty} b q^n f(b q^n).$$

If $f : [0..b] \to \mathbf{R}$ is continuous then

$$\lim_{q \to 1} \int_0^b f(t)\, d_q t = \int_0^b f(t)\, dt.$$

Jackson also defined the improper q-integrals

$$\int_0^{\infty} f(t)\, d_q t := (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^{\infty} f(t)\, d_q t := (1-q) \sum_{n=-\infty}^{\infty} q^n \big[f(q^n) + f(-q^n) \big].$$

Fundamental Theorem of q-Calculus

Suppose $f : (0, b] \to \mathbf{R}$ and $0 < x \le b$.

The Jackson q-integral of f is defined by

$$\int_0^x f(t)\, d_q t := (1-q) \sum_{j=0}^{\infty} x q^j f(x q^j).$$

If there exists $0 \le \alpha < 1$ such that $|f(t)t^\alpha|$ is bounded on $(0, b]$, then the integral converges to a function $F(x)$ on $(0, b]$.

Additionally, F is a q-antiderivative of f:

$$D_q F(x) := \frac{F(qx) - F(x)}{(q-1)x} = f(x), \quad 0 < x \le b.$$

Note that

$$\lim_{q \to 1} D_q F(x) = F'(x),$$

and

$$\lim_{q \to 1} \int_0^x f(t)\, d_q t = \int_0^x f(t)\, dt.$$

The q-analogue of α

Recall that if $y = f(x)$, then

$$\frac{d_q\, y}{d_q\, x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Example. Let $\alpha \in \mathbf{R}$ and $y = x^\alpha$. Then

$$\frac{d_q\, y}{d_q\, x} = \frac{(qx)^\alpha - x^\alpha}{(q-1)x} = \left(\frac{q^\alpha - 1}{q-1}\right) x^{\alpha-1},$$

$$\lim_{q \to 1} \frac{d_q\, y}{d_q x} = \frac{dy}{dx} = \alpha\, x^{\alpha-1}.$$

Definition. Let $\alpha \in \mathbf{R}$ and $q \ne 1$. The q-*analogue* of α is

$$[\alpha]_q := \frac{q^\alpha - 1}{q-1}.$$

The q-Factorial

Define the q-factorial of a positive integer n by

$$[n]!_q := \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q} = \prod_{k=1}^{n} (1 + q + \cdots + q^{k-1}).$$

Let $\mathfrak{S}_n$ denote the set of permutations on $\{1, 2, \ldots, n\}$.

Definition. An *inversion* in a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i, j) with $i < j$ and such that $\sigma(i) > \sigma(j)$.

If we expand $[n]!_q$ in powers of q:

$$[n]!_q = \sum_{k=0}^{n(n-1)/2} a_k\, q^k,$$

then

$$a_k = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } k \text{ inversions}\}.$$

Inversions in Permutations for $\mathfrak{S}_3$

Definition. An *inversion* in a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i, j) with $i < j$ and such that $\sigma(i) > \sigma(j)$.

permutation	123	132	213	231	312	321
# inversions	0	1	1	2	2	3

If we expand $[n]!_q$ in powers of q:

$$[n]!_q = \prod_{k=1}^{n} (1 + q + \cdots + q^{k-1}) = \sum_{k=0}^{n(n-1)/2} a_k\, q^k,$$

then

$$a_k = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } k \text{ inversions}\}.$$

$$[3]!_q = (1+q)(1+q+q^2) = 1 + 2q + 2q^2 + q^3.$$

$$a_0 = a_3 = 1, a_1 = a_2 = 2.$$

q-Binomial Coefficients

Recall the q-factorial:

$$[n]!_q := \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q}.$$

For $0 \le k \le n$, the q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

Note that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

We also have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{1-q^{n-j+1}}{1-q^j}.$$

Theorem 1 (G.-C. Rota) *Let q be a power of a prime positive integer. Then $\begin{bmatrix} n \\ k \end{bmatrix}$ is equal to the number of k-dimensional subspaces of the n-dimensional vector space*

$$\mathbf{F}_q^{\oplus n} = \underbrace{\mathbf{F}_q \oplus \cdots \oplus \mathbf{F}_q}_{n},$$

where $\mathbf{F}_q$ is the finite field with q elements.

Proof (Sketch). Fix $1 \le k \le n$. Any k-dimensional subspace is determined by a basis consisting of k linearly independent spanning vectors.

There are $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$ bases for all k-dimensional subspaces.

Since different bases may span the same subspace, we divide by the number of possible choices of basis of a particular k-dimensional subspace:

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}).$$

The quotient is $\begin{bmatrix} n \\ k \end{bmatrix}$. □

q-Powers

Let $0 \le n \in \mathbf{Z}$ and $x, y \in \mathbf{R}$.

Define the asymmetric q-power by

$$(x+y)_q^n := \prod_{k=0}^{n-1} (x + yq^k).$$

Clearly,

$$\lim_{q \to 1} (x+y)_q^n = (x+y)^n.$$

Theorem 2 (Gauss Finite q-Binomial) *Let $x, y \in \mathbf{R}$ and $0 \le n \in \mathbf{Z}$. Then*

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k.$$

Letting $q \to 1$, we deduce

Corollary 1 (Classical Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Recall that for $x, y \in \mathbf{R}$ and $0 \le n \in \mathbf{Z}$,

$$(x+y)_q^n := \prod_{k=0}^{n-1} (x + yq^k), \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{1-q^{n-j+1}}{1-q^j}.$$

Theorem 2 (Gauss Finite q-Binomial)

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k.$$

Corollary 2 (q-Vandermonde Convolution)

$$\begin{bmatrix} a+b \\ n \end{bmatrix} = \sum_{k=0}^{n} q^{(n-k)(a-k)} \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ n-k \end{bmatrix}.$$

Proof Sketch. Compare coefficients of y^n in

$$(1+y)_q^{a+b} = (1+y)_q^a \, (1+yq^a)_q^b.$$

□

Remark. The limiting ($q \to 1$) case of the q-Vandermonde Convolution is sometimes referred to as the *Hypergeometric Identity*.

Heine-Roth Implies Gauss

Let $\alpha \in \mathbf{R}$, $z \in \mathbf{C}$. Define

$$(1+z)_q^\alpha := \frac{(1+z)_q^\infty}{(1+zq^\alpha)_q^\infty} = \frac{(-z;q)_\infty}{(-zq^\alpha;q)_\infty}.$$

For $0 \le k \in \mathbf{Z}$ define

$$\left[\begin{array}{c}\alpha\\k\end{array}\right] := \prod_{j=1}^{k} \frac{1-q^{\alpha-j+1}}{1-q^j}.$$

By the Heine-Roth q-binomial theorem,

$$(1+z)_q^\alpha = \sum_{k=0}^{\infty} \frac{(q^{-\alpha};q)_k}{(q;q)_k}(-zq^\alpha)^k.$$

But

$$\frac{(q^{-\alpha};q)_k}{(q;q)_k}(-q^\alpha)^k = q^{k(k-1)/2}\left[\begin{array}{c}\alpha\\k\end{array}\right].$$

Therefore,

$$(1+z)_q^\alpha = \sum_{k=0}^{\infty} q^{k(k-1)/2}\left[\begin{array}{c}\alpha\\k\end{array}\right] z^k.$$

The q-Exponential Function

If $0 < q < 1$, then

$$\lim_{n\to\infty}\left[\begin{array}{c}n\\k\end{array}\right] = \lim_{n\to\infty}\prod_{j=1}^{k}\frac{1-q^{n-k+j}}{1-q^j} = \prod_{j=1}^{k}\frac{1}{1-q^j} = \frac{1}{(1-q)^k\,[k]!_q}.$$

Recall the q-binomial theorem in the form

$$(1+x)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2}\left[\begin{array}{c}n\\k\end{array}\right]x^k, \quad 0 \le n \in \mathbf{Z}.$$

Thus,

$$\prod_{j=0}^{\infty}(1+xq^j) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{[k]!_q}\left(\frac{x}{1-q}\right)^k.$$

Define the q-exponential function by

$$\exp_q(x) := \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}\,x^k}{[k]!_q} = (1+(1-q)x)_q^\infty.$$

Then

$$\lim_{q\to 1}\exp_q(x) = e^x, \quad D_q\exp_q(x) = \exp_q(qx).$$

Multiple Zeta Values

$$\zeta(s_1,\dots,s_m) := \sum_{k_1>\cdots>k_m>0}\ \prod_{j=1}^{m} k_j^{-s_j}.$$

The multiple series is absolutely convergent if

$$\sum_{j=1}^{n}\Re(s_j) > n, \qquad n = 1,2,\dots,m.$$

Euler ($m=2$):

$$2\zeta(s,1) = s\zeta(s+1) - \sum_{j=1}^{s-2}\zeta(s-j)\zeta(j+1),$$

where $2 \le s \in \mathbf{Z}$.

Period One

For all non-negative integers n,

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\{4\}^n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!},$$

$$\zeta(\{6\}^n) = \frac{6\cdot(2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(\{8\}^n) = \frac{8\cdot(2\pi)^{8n}}{(8n+4)!}, \times\left\{\left(1+\frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1-\frac{1}{\sqrt{2}}\right)^{4n+2}\right\}.$$

More generally, let $k \in \mathbf{Z}^+$ and $\omega := e^{i\pi/k}$. Then

$$\sum_{n=0}^{\infty}(-1)^n x^{2kn}\zeta(\{2k\}^n) = \prod_{j=0}^{k-1}\frac{\sin(\pi x\omega^j)}{\pi x\omega^j}.$$

Period Two

For all non-negative integers n,

$$\zeta(\{3,1\}^n) = 4^{-n}\zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!},$$

$$\zeta(3,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n}\zeta(4k+3)\zeta(\{4\}^{n-k}) = \sum_{k=0}^{n}\frac{2\pi^{4k}}{(4k+2)!}\left(-\frac{1}{4}\right)^{n-k}\zeta(4n-4k+3),$$

$$\zeta(2,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n}(-1)^k\zeta(\{4\}^{n-k})\Big\{(4k+1) \times \zeta(4k+2) - 4\sum_{j=1}^{k}\zeta(4j-1)\zeta(4k-4j+3)\Big\}.$$

Multiple q-Zeta Values

M. Kaneko investigated analytic properties of the Riemann q-zeta function

$$\zeta[s] := \sum_{k=1}^{\infty}\frac{q^{tk}}{[k]_q^s}, \qquad t = s-1.$$

This suggests that we consider

$$\zeta[s_1,\dots,s_m] := \sum_{k_1>\cdots>k_m>0}\prod_{j=1}^{m}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Alternatively, we could let

$$Z[s_1,\dots,s_m] := \sum_{k_1>\cdots>k_m>0}\prod_{j=1}^{m}\frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

If $\vec{s} = (s_1,\dots,s_m)$, then

$$\zeta[\vec{s};q] = q^{|\vec{s}|}Z[\vec{s};1/q], \qquad |\vec{s}| := \sum_{j=1}^{m}s_j.$$

Period-1 Sums Reduce

Theorem 3 *If n is a positive integer and $s>1$, then*

$$n\zeta[\{s\}^n] = \sum_{k=1}^{n}(-1)^{k+1}\zeta[\{s\}^{n-k}]\sum_{j=0}^{k-1}\binom{k-1}{j}(1-q)^j\zeta[ks-j].$$

Example 1 *With $n=2$, we get*

$$2\zeta[s,s] = \zeta[s]\zeta[s] - \Big(\zeta[2s] + (1-q)\zeta[2s-1]\Big).$$

Corollary 3 *If n is a positive integer and $s>1$, then*

$$n\zeta(\{s\}^n) = \sum_{k=1}^{n}(-1)^{k+1}\zeta(\{s\}^{n-k})\zeta(ks).$$

Let $\mathfrak{S}_n$ denote the group of $n!$ permutations of $\langle n\rangle = \{1,2,\dots,n\}$.

Theorem 4 *Let n be a positive integer, and let $s_j>1$ for $1\le j\le n$. Then*

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\Big[\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\Big] = \sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{k=1}^{|\mathcal{P}|}(|P_k|-1)! \times \sum_{\nu_k=0}^{|P_k|-1}\binom{|P_k|-1}{\nu_k}(1-q)^{\nu_k}\zeta[p_k-\nu_k],$$

where the outer sum on the right is over all unordered set partitions $\mathcal{P} = \{P_1,\dots,P_m\}$ of $\langle n\rangle$, $1\le m=|\mathcal{P}|\le n$, and $p_k = \sum_{j\in P_k}s_j$.

Corollary 4 (M. Hoffman)

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\Big(\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\Big) = \sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{P\in\mathcal{P}}(|P|-1)!\,\zeta\Big(\sum_{j\in P}s_j\Big).$$

Parity Reduction

Theorem 5 *Let $m \in \mathbf{Z}^+$ and let $s_1, \ldots, s_m$ be real numbers with $s_1 > 1$, $s_m > 1$, and $s_j \geq 1$ for $1 < j < m$. Then*

$$\zeta\Big[\operatorname*{Cat}_{k=1}^{m} s_k\Big] + (-1)^m \zeta\Big[\operatorname*{Cat}_{k=1}^{m} s_{m-k+1}\Big]$$

can be expressed as a $\mathbf{Z}[q]$-linear combination of multiple q-zeta values of depth less than m.

That is, the coefficients in the linear combination are polynomials in q with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

A Double Generating Function

Theorem 6

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^{m+1}v^{n+1}\zeta[m+2,\{1\}^n]$$
$$= 1 - \exp\left\{\sum_{k=2}^{\infty}\left\{u^k + v^k - \Big(u+v+(1-q)uv\Big)^k\right\} \times \frac{1}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\}.$$

Corollary 5 *If $0 \leq m, n \in \mathbf{Z}$, then*

$$\zeta[m+2,\{1\}^n] = \zeta[n+2,\{1\}^m].$$

Corollary 6 (q-Euler) *Let $0 \leq m \in \mathbf{Z}$. Then*

$$2\zeta[m+2,1] = (m+2)\zeta[m+3] + (1-q)\, m\, \zeta[m+2] - \sum_{k=2}^{m+1}\zeta[m+3-k]\,\zeta[k].$$

In particular, when $m = 0$ we get $\zeta[2,1] = \zeta[3]$.

The proof of Theorem 6 makes essential use of the basic hypergeometric function

$${}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}\,\middle|\, x\right] = \sum_{n=0}^{\infty}\frac{(1-q^a)_q^n(1-q^b)_q^n}{(1-q^c)_q^n(1-q)_q^n}\,x^n, \qquad |x| < 1.$$

Routine series manipulations reveal that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta[m+2,\{1\}^n] = 1 - {}_2\phi_1\left[\begin{matrix} q^{-y}, q^x \\ q^{1+x} \end{matrix}\,\middle|\, q^{1+y}\right]$$

Heine's q-analog

$${}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}\,\middle|\, q^{c-a-b}\right] = \frac{\Gamma_q(c)\Gamma_q(c-a-b)}{\Gamma_q(c-a)\Gamma_q(c-b)}$$

of Gauss's ${}_2F_1$ summation formula then gives

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta[m+2,\{1\}^n] = 1 - \frac{\Gamma_q(1+x)\Gamma_q(1+y)}{\Gamma_q(1+x+y)}.$$

If $\vec{s} = (s_1, \ldots, s_m)$ define

$$\text{weight}(\vec{s}) := |\vec{s}| = \sum_{j=1}^{m} s_j,$$
$$\text{depth}(\vec{s}) := m,$$
$$\text{height}(\vec{s}) := \#\{j : s_j \geq 2\}.$$

Theorem 7 (J. Okuda & Y. Takeyama)

$$1 + (w-uv)\sum_{s,m,h\geq 0} u^{s-m-h}v^{m-h}w^{h-1}\sum_{\substack{\text{weight}(\vec{s})=s\\ \text{depth}(\vec{s})=m\\ \text{height}(\vec{s})=h}}\zeta[\vec{s}]$$
$$= \exp\left\{\sum_{k=2}^{\infty}\Big(u^k + v^k - \alpha^k - \beta^k\Big)\frac{1}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\},$$

where α and β satisfy the equations

$$\alpha + \beta = u + v + (q-1)(w-uv), \qquad \alpha\beta = w.$$

Theorem 6 is case $w = 0$ of Theorem 7.

The Simplex Integral

M. Kontsevich: If $s_1, \ldots, s_m \in \mathbf{Z}^+$, then

$$\zeta(s_1, \ldots, s_m) = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0,$$

and is abbreviated (D. Broadhurst) by

$$\int_0^1 \prod_{k=1}^{m} A^{s_k - 1} B, \qquad A = \frac{dt}{t}, \quad B = \frac{dt}{1-t}.$$

$$\zeta(2,1) = \sum_{n>m>0} n^{-2} m^{-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k+j)^{-2} k^{-1}$$

$$= \sum_{k=1}^{\infty} k^{-1} \sum_{j=1}^{\infty} (k+j)^{-1} \int_0^1 t^{k+j-1}\, dt$$

$$= \sum_{k=1}^{\infty} k^{-1} \int_0^1 t^{-1} \sum_{j=1}^{\infty} \int_0^t u^{k+j-1}\, du\, dt$$

$$= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} k^{-1} u^k\, du\, dt$$

$$= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} \int_0^u v^{k-1}\, dv\, du\, dt$$

$$= \int_{1>t>u>v>0} \frac{dt}{t} \cdot \frac{du}{1-u} \cdot \frac{dv}{1-v}$$

$$= \int_0^1 AB^2.$$

The Jackson Integral

Suppose $f : (0, b] \to \mathbf{R}$ and $0 < x \le b$.

The Jackson q-integral of f on the subinterval $(0, x]$ is

$$\int_0^x f(t)\, d_q t := (1-q) \sum_{j=0}^{\infty} x q^j f(x q^j).$$

If there exists $0 \le \alpha < 1$ such that $|f(t) t^{\alpha}|$ is bounded on $(0, b]$, then the integral converges to a function $F(x)$ on $(0, b]$.

Additionally, F is a q-antiderivative of f:

$$D_q F(x) := \frac{F(qx) - F(x)}{(q-1)x} = f(x), \quad 0 < x \le b.$$

Note that

$$\lim_{q \to 1} D_q F(x) = F'(x),$$

and

$$\lim_{q \to 1} \int_0^x f(t)\, d_q t = \int_0^x f(t)\, dt.$$

The Jackson Simplex Integral

Let $s_1, \ldots, s_m$ are positive integers. Recall:

$$\zeta(s_1, \ldots, s_m) = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0.$$

Theorem 8

$$\zeta[s_1, \ldots, s_m] = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{d_q t_r^{(k)}}{t_r^{(k)}} \right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}},$$

where

$$y_k := \prod_{j=1}^{k} q^{1 - s_j},$$

and the integral is over the same simplex as above.

Duality

Let $a_i, b_i \in \mathbf{Z}^+$ and $k = \sum_{i=1}^{n}(a_i + b_i)$. Then

$$\zeta(a_1+1, \{1\}^{b_1-1}, \dots, a_n+1, \{1\}^{b_n-1})$$

$$= \int_0^1 \prod_{i=1}^{n} A^{a_i} B^{b_i}$$

$$= \int_{1>t_1>\cdots>t_k>0} \prod_{j=1}^{k} f_j(t_j)\, dt_j$$

$$= \int_{1>u_k>\cdots>u_1>0} \prod_{j=1}^{k} f_j(u_j)\, du_j, \quad u_j = 1 - t_j$$

$$= \int_0^1 \prod_{i=n}^{1} A^{b_i} B^{a_i}$$

$$= \zeta(b_n+1, \{1\}^{a_n-1}, \dots, b_1+1, \{1\}^{a_1-1}).$$

Generalized Duality

Definition 1 *Let n and $s_1, \dots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S(s_1, \dots, s_n; m) := \sum_{\substack{c_1,\dots,c_n \ge 0 \\ c_1+\cdots+c_n = m}} \zeta(s_1 + c_1, \dots, s_n + c_n).$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \{a_i + 1, \{1\}^{b_i-1}\},$$
$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \{b_i + 1, \{1\}^{a_i-1}\}.$$

Theorem 9 (Y. Ohno) *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have the equality*

$$S(\vec{s}; m) = S(\vec{s}\,'; m).$$

Generalized q-Duality

Definition 2 *Let n and $s_1, \dots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S[s_1, \dots, s_n; m] := \sum_{\substack{c_1,\dots,c_n \ge 0 \\ c_1+\cdots+c_n = m}} \zeta[s_1 + c_1, \dots, s_n + c_n].$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \{a_i + 1, \{1\}^{b_i-1}\}$$
$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \{b_i + 1, \{1\}^{a_i-1}\}.$$

Theorem 10 *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have*

$$S[\vec{s}; m] = S[\vec{s}\,'; m].$$

q-Duality

Corollary 7 *If $\vec{s}$, $\vec{s}\,'$ are dual argument lists, then*

$$\zeta[\vec{s}] = \zeta[\vec{s}\,'].$$

In other words, if $a_i, b_i \in \mathbf{Z}^+$ $(1 \le i \le n)$, then

$$\zeta\Big[\operatorname*{Cat}_{i=1}^{n} \{a_i + 1, \{1\}^{b_i-1}\}\Big] = \zeta\Big[\operatorname*{Cat}_{i=n}^{1} \{b_i + 1, \{1\}^{a_i-1}\}\Big].$$

Proof. Put $m = 0$ in Theorem 10 (generalized q-duality). □

q-Sum Formula

Definition 3 *Let $t_1, \ldots, t_n$ be positive integers.*

$$\zeta^*[t_1, \ldots, t_n] := \zeta\Big[t_1 + 1, \operatorname*{Cat}_{j=2}^{n} t_j\Big].$$

Corollary 8 (q-Sum Formula) *For any integers $0 < k \le n$, we have*

$$\sum_{t_1+t_2+\cdots+t_n=k} \zeta^*[t_1, t_2, \ldots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers $t_1, \ldots, t_n$ with sum equal to k.

Proof. If we take the dual argument lists in the form $\vec{s} = (n+1)$ and $\vec{s}\,' = (2, \{1\}^{n-1})$ and put $m = k - n$, then Theorem 10 states that

$$\zeta[k+1] = \sum_{\substack{c_1,\ldots,c_n \ge 0 \\ c_1+\cdots+c_n=k-n}} \zeta\Big[2 + c_2, \operatorname*{Cat}_{j=2}^{n}\{1 + c_j\}\Big]$$

$$= \sum_{\substack{t_1,\ldots,t_n \ge 1 \\ t_1+\cdots+t_n=k}} \zeta\Big[t_1 + 1, \operatorname*{Cat}_{j=2}^{n} t_j\Big].$$

□

q-Cyclic Sum Formula

Definition 4 *Let $s_j \in \mathbf{Z}^+$ for $1 \le j \le n$ and put $\vec{s} = (s_1, \ldots, s_n)$. Let σ denote the n-cycle $(1\ 2 \cdots n)$, and let*

$$\mathcal{C}(\vec{s}) := \{(s_{\sigma^j(1)}, \ldots, s_{\sigma^j(n)}) : 1 \le j \le n\}$$

denote the set of cyclic permutations of $\vec{s}$.

Recall the definition

$$\zeta^*[s_1, \ldots, s_n] := \zeta[s_1 + 1, s_2, \ldots, s_n].$$

Theorem 11 *Let $\vec{s}$ and $\vec{s}\,'$ be dual argument lists. Then*

$$\sum_{\vec{t} \in \mathcal{C}(\vec{s})} \zeta^*[\vec{t}\,] = \sum_{\vec{t} \in \mathcal{C}(\vec{s}\,')} \zeta^*[\vec{t}\,].$$

Proof of Generalized q-Duality

Let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y.

Let $\mathfrak{h}^0$ denote the subalgebra $\mathbf{Q}1 \oplus x\mathfrak{h}y$.

The $\mathbf{Q}$-linear map $\widehat{\zeta}$ is defined on $\mathfrak{h}^0$ by

$$\widehat{\zeta}[1] := \zeta[\,] = 1$$

and

$$\widehat{\zeta}\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}\right] = \zeta\left[\operatorname*{Cat}_{i=1}^{s}\left\{a_i + 1, \{1\}^{b_i - 1}\right\}\right],$$

for positive integers a_i, b_i $(1 \le i \le s)$.

Let τ be the anti-automorphism of $\mathfrak{h}$ that switches x and y.

Then q-duality simply says that

$$\widehat{\zeta}[\tau w] = \widehat{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.$$

For each $n \in \mathbf{Z}^+$, let D_n be the derivation on $\mathfrak{h}$ that maps $x \mapsto 0$ and $y \mapsto x^n y$.

Let θ be an indeterminate (formal parameter).

Define

$$\Delta := \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n \quad \text{and} \quad \sigma := \exp(\Delta).$$

Then

Δ is a derivation on $\mathfrak{h}[[\theta]]$, and
σ is an automorphism of $\mathfrak{h}[[\theta]]$.

For any word $w \in \mathfrak{h}^0$, define

$$f[w;\theta] := \widehat{\zeta}[\sigma w]$$

and

$$g[w;\theta] := f[\tau w;\theta] = \widehat{\zeta}[\sigma\tau w].$$

Recall

$$D_n \text{ sends } x \mapsto 0,\ y \mapsto x^n y.$$

Thus,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n : x \mapsto 0,\ y \mapsto \{\log(1 - x\theta)^{-1}\} y$$

and

$$\sigma = \exp(\Delta) : x \mapsto x,\ y \mapsto (1 - x\theta)^{-1} y.$$

Therefore,

$$f\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta\Big] = \widehat{\zeta}\Big[\prod_{i=1}^{s} x^{a_i} \big\{(1 - x\theta)^{-1} y\big\}^{b_i}\Big]$$

$$= \sum_{m=0}^{\infty} \theta^m \sum_{\substack{c_1,\ldots,c_n \geq 0 \\ c_1 + \cdots + c_n = m}} \zeta\Big[\operatorname*{Cat}_{j=1}^{n}\{t_j + c_j\}\Big],$$

$$= \sum_{m=0}^{\infty} \theta^m S[\vec{t}; m],$$

where

$$\vec{t} = (t_1, \ldots, t_n) = (\operatorname*{Cat}_{i=1}^{s}\{a_i + 1, \{1\}^{b_i - 1}\}), n = \sum_{i=1}^{s} b_i.$$

Generalized q-Duality Reformulated

Theorem 12 *For all $w \in \mathfrak{h}^0$, $f[w; \theta] = g[w; \theta]$, i.e. $\widehat{\zeta} \circ \sigma$ is invariant under ordinary duality τ.*

Theorem 13 *Let $a_i, b_i \in \mathbf{Z}^+$ and $\sum_{i=1}^{s}(a_i + b_i) > 2$. Let $\theta' := q\theta - 1$, and set*

$$I^m = \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{m}.$$

The generating functions f and g satisfy the difference equation

$$\sum_{\substack{\epsilon,\delta \in I^s \\ \delta_1 < a_1, \epsilon_s < b_s}} (-\theta)^{\overline{\delta}\cdot\overline{\epsilon}} (1-q)^{\delta\cdot\epsilon} f\Big[\prod_{i=1}^{s} x^{a_i - \delta_i} y^{b_i - \epsilon_i}; \theta\Big]$$

$$= \sum_{\substack{\delta,\epsilon \in I^{s+1} \\ \delta_{s+1} = \epsilon_1 = 0 \\ \delta_1 < a_1, \epsilon_{s+1} < b_s}} (-\theta')^{\overline{\delta}\cdot\overline{\epsilon} - 1} (1-q)^{\delta\cdot\epsilon} f\Big[\prod_{i=1}^{s} x^{a_i - \delta_i} y^{b_i - \epsilon_{i+1}}; \theta'\Big].$$

Here, $\overline{\delta}$ to denotes the ordered tuple whose i^{th} component is $1 - \delta_i$, and of course $\delta \cdot \epsilon$ denotes the dot product $\sum_i \delta_i \epsilon_i$. Similarly, $\overline{\epsilon}$ denotes the ordered tuple whose i^{th} component is $1 - \epsilon_i$, and $\overline{\delta}\cdot\overline{\epsilon} = \sum_i (1 - \delta_i)(1 - \epsilon_i)$.

Proof of Theorem 12. Use induction on the total degree of the word $\prod_{i=1}^{s} x^{a_i} y^{b_i}$.

The base case is clearly satisfied, since the word xy is self-dual.

Now apply Theorem 13 to f and g and subtract the resulting two equations.

The terms whose words have total degree less than $\sum_{i=1}^{s}(a_i + b_i)$ are cancelled by the induction hypothesis.

This leaves us with

$$(-\theta)^s \Big\{ f\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta\Big] - g\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta\Big]\Big\}$$

$$= (-\theta')^s \Big\{ f\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta'\Big] - g\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta'\Big]\Big\}.$$

Thus, the function

$$H(\theta) := (-\theta)^s \Big\{ f\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta\Big] - g\Big[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta\Big]\Big\}$$

satisfies the functional equation $H(\theta) = H(\theta')$, where $\theta' = q\theta - 1$.

One can show that $H(\theta)$ is a meromorphic function of θ of the form

$$H(\theta) = \theta^s \sum_{n=1}^{\infty} \frac{h_n}{[n]_q - \theta q^n},$$

with at worst simple poles at $\theta = p_n := q^{-n}[n]_q$ for positive integers n.

Note that

$$0 = p_0 < p_1 < p_2 < \cdots < p_{n-1} < p_n < \cdots$$

and

$$p_n{}' = q p_n - 1 = p_{n-1} \quad \text{for all } n \geq 1.$$

The functional equation

$$H(\theta) = H(\theta')$$

thus implies that if H has a pole at p_n, then H must also have a pole at p_{n-1}.

Since H has no pole at p_0, it follows that each $h_n = 0$.

Thus, H vanishes identically and so $f = g$. □

Derivations

Definition 5 (K. Ihara & M. Kaneko) *Define a derivation on $\mathfrak{h}$ for each positive integer n by*

$$\partial_n(x) = x(x+y)^{n-1}, \qquad \partial_n(y) = -x(x+y)^{n-1}y.$$

Theorem 14 (Ihara & Kaneko) *For all positive integers n and words $w \in \mathfrak{h}^0$, $\widehat{\zeta}(\partial_n(w)) = 0$.*

Theorem 15 (q-Analog) *For all positive integers n and words $w \in \mathfrak{h}^0$, $\widehat{\zeta}[\partial_n(w)] = 0$.*

Theorem 15 is actually *equivalent* to generalized q-duality (Theorem 12).

Proof of Theorem 15

Proof. Let $\sigma = \exp(\Delta)$, $\widetilde{\sigma} = \tau\sigma\tau$,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n}\theta^n, \qquad \partial = \sum_{n=1}^{\infty} \frac{\partial_n}{n}\theta^n.$$

Generalized q-duality (Theorem 12): $\forall w \in \mathfrak{h}^0$,

$$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w] = \widehat{\zeta}[\tau\sigma\tau w] \iff (\sigma - \widetilde{\sigma})w \in \ker\widehat{\zeta}.$$

We show that in fact, $(\sigma - \widetilde{\sigma})\mathfrak{h}^0 = \partial\mathfrak{h}^0$.

To prove this, we require the following identity of Ihara and Kaneko.

Proposition 16 $\exp(\partial) = \widetilde{\sigma}\sigma^{-1}$.

To complete the proof of Theorem 15, observe that since

$$\partial = \log\left(\widetilde{\sigma}\sigma^{-1}\right) = \log\left(1 - (\sigma - \widetilde{\sigma})\sigma^{-1}\right)$$

$$= -(\sigma - \widetilde{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n}\left((\sigma - \widetilde{\sigma})\sigma^{-1}\right)^{n-1}\sigma^{-1},$$

and

$$\sigma - \widetilde{\sigma} = \left(1 - \widetilde{\sigma}\sigma^{-1}\right)\sigma = \left(1 - \exp(\partial)\right)\sigma$$
$$= -\partial \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{n!}\sigma,$$

we see that

$$\partial\mathfrak{h}^0 \subseteq (\sigma - \widetilde{\sigma})\mathfrak{h}^0 \quad \text{and} \quad (\sigma - \widetilde{\sigma})\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0.$$

Thus for the kernel of $\widehat{\zeta}$, we have the equivalences

$$(\sigma - \widetilde{\sigma})w \in \ker\widehat{\zeta} \iff \partial w \in \ker\widehat{\zeta}$$
$$\iff \forall n \in \mathbf{Z}^+,\ \widehat{\zeta}[\partial_n w] = 0.$$

□