

# A $q$ -analog of Euler's reduction formula for the double zeta function

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# Double Zeta Function

**Definition 1.** The Riemann zeta function is defined in the right half-plane  $\{s \in \mathbb{C} : \Re(s) > 1\}$  by the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

**Definition 2.** The double zeta function is a function of two complex variables defined by the double series

$$\zeta(s, t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \quad \Re(s) > 1, \quad \Re(s + t) > 2.$$

The double zeta function was first studied by Euler in response to a letter from Goldbach in 1742.

**Theorem 1 (Euler)** *If  $s - 1$  and  $t - 1$  are positive integers, then the decomposition formula*

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a)$$

*holds.*

A combinatorial proof of Euler's decomposition formula based on the simplex integral representations

$$\zeta(s) = \int_{1 > x_1 > \dots > x_s > 0} \left( \prod_{i=1}^{s-1} \frac{dx_i}{x_i} \right) \frac{dx_s}{1-x_s},$$

$$\zeta(s, t) = \int_{1 > x_1 > \dots > x_{s+t} > 0} \left( \prod_{i=1}^{s-1} \frac{dx_i}{x_i} \right) \frac{dx_s}{1-x_s} \left( \prod_{i=s+1}^{s+t-1} \frac{dx_i}{x_i} \right) \frac{dx_{s+t}}{1-x_{s+t}},$$

and the shuffle multiplication rule satisfied by such integrals is given in a 1998 EJC paper of J. Borwein, D. Bradley, D. Broadhurst and P. Lisoněk.

# Euler's Decomposition Formula a.k.a. Shuffle Product

Previously, in their paper

Explicit Evaluation of Euler Sums, *Proc. Edinburgh Math. Soc.* **38** (1995), 277-294

D. Borwein, J. Borwein and R. Girgensohn gave an algebraic (integral-free) proof of Euler's decomposition formula (for integers  $s > 1$ ,  $t > 1$ )

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a)$$

by summing Nielsen's partial fraction decomposition

$$\frac{1}{x^s(c-x)^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a}c^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{c^{s+a}(c-x)^{t-a}}$$

over appropriately chosen integers  $x$  and  $c$ .

# Euler's Reduction Formula

In his 1742 letter to Euler, Goldbach actually proposed the problem of evaluating  $\zeta(s, t)$  for integers  $s > 1$  and  $t > 0$ .

Calculating several examples led Euler to infer a closed form evaluation in terms of values of the Riemann zeta function, in the case when the two arguments have opposite parity:

Let  $s - 1$  and  $t - 1$  be positive integers with  $s + t$  odd and let  $2h = \max(s, t)$ . Then the reduction formula

$$\zeta(s, t) = \frac{1}{2} \left( (1 + (-1)^s) \zeta(s) \zeta(t) + \frac{1}{2} \left[ (-1)^s \binom{s+t}{s} - 1 \right] \zeta(s+t) \right) \\ + (-1)^{s+1} \sum_{k=1}^h \left[ \binom{s+t-2k-1}{t-1} + \binom{s+t-2k-1}{s-1} \right] \zeta(2k) \zeta(s+t-2k)$$

holds.

Euler apparently did not give a proof of his reduction formula.

A proof was later given by D. Borwein, J. Borwein and R. Girgensohn in their 1995 *Proc. Edinburgh Math. Soc.* paper, Explicit Evaluation of Euler Sums.

Using the reflection formula (a.k.a. stuffle product)

$$\zeta(s)\zeta(t) = \zeta(s, t) + \zeta(s + t) + \zeta(t, s)$$

obtained by grouping the indices  $(n, k)$  in the series on the left according to whether  $n > k$ ,  $n = k$  or  $n < k$ , and Euler's decomposition formula

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a),$$

they derived a system of linear equations for the values of  $\zeta(s, t)$  with  $s + t$  fixed but odd, and then solved the resulting system.

When  $s + t$  is even, the system does not in general have a unique solution.

# Degenerate Case: Euler's Convolution Formula

Recall that Euler's reduction formula

$$\zeta(s, t) = \frac{1}{2}((1 + (-1)^s)\zeta(s)\zeta(t) + \frac{1}{2}\left[(-1)^s \binom{s+t}{s} - 1\right]\zeta(s+t) \\ + (-1)^{s+1} \sum_{k=1}^h \left[ \binom{s+t-2k-1}{t-1} + \binom{s+t-2k-1}{s-1} \right] \zeta(2k)\zeta(s+t-2k),$$

is valid for integers  $s > 1$  and  $t > 1$  with  $s + t$  odd and  $2h = \max(s, t)$ .

If we interpret  $\zeta(1) = 0$  the formula gives true results also when  $t = 1$  and  $s$  is even, but this case is subsumed by another formula of Euler, namely the convolution formula

$$\zeta(s, 1) = \frac{1}{2}s \zeta(s+1) - \frac{1}{2} \sum_{k=2}^{s-1} \zeta(k)\zeta(s+1-k),$$

which is valid for all integers  $s > 1$ , not just even  $s > 1$ .

# Multiple Zeta Values

**Definition.** (Michael Hoffman 1992, Don Zagier 1994)

$$\zeta(s_1, \dots, s_m) := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m k_j^{-s_j}.$$

The multiple series is absolutely convergent if

$$\sum_{j=1}^n \Re(s_j) > n, \quad n = 1, 2, \dots, m.$$

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**Example.** With  $m = 2$ ,  $s_1 = s$  and  $s_2 = 1$ , we have Euler's convolution formula

$$2\zeta(s, 1) = s\zeta(s+1) - \sum_{j=1}^{s-2} \zeta(s-j)\zeta(j+1),$$

where  $2 \leq s \in \mathbf{Z}$ . In particular,  $\zeta(2, 1) = \zeta(3)$ .



# Period One

For all non-negative integers  $n$ ,

$$\zeta(\{2\}^n) := \zeta(\underbrace{2, 2, \dots, 2}_n) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\{4\}^n) := \zeta(\underbrace{4, 4, \dots, 4}_n) = \frac{2^{2n+1} \pi^{4n}}{(4n+2)!},$$

$$\zeta(\{6\}^n) := \zeta(\underbrace{6, 6, \dots, 6}_n) = \frac{6(2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(\{8\}^n) := \zeta(\underbrace{8, 8, \dots, 8}_n) = \frac{8(2\pi)^{8n}}{(8n+4)!} \left\{ \left(1 + \frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1 - \frac{1}{\sqrt{2}}\right)^{4n+2} \right\}.$$

More generally, let  $k \in \mathbf{Z}^+$  and  $\omega := e^{i\pi/k}$ . Then

$$\sum_{n=0}^{\infty} (-1)^n x^{2kn} \zeta(\{2k\}^n) = \prod_{j=0}^{k-1} \frac{\sin(\pi x \omega^j)}{\pi x \omega^j}.$$

# Period Two

For all non-negative integers  $n$ ,

$$\zeta(\{3, 1\}^n) = 4^{-n} \zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!},$$

$$\begin{aligned} \zeta(3, \{1, 3\}^n) &= 4^{-n} \sum_{k=0}^n \zeta(4k+3) \zeta(\{4\}^{n-k}) \\ &= \sum_{k=0}^n \frac{2\pi^{4k}}{(4k+2)!} \left(-\frac{1}{4}\right)^{n-k} \zeta(4n-4k+3), \end{aligned}$$

$$\begin{aligned} \zeta(2, \{1, 3\}^n) &= 4^{-n} \sum_{k=0}^n (-1)^k \zeta(\{4\}^{n-k}) \left\{ (4k+1) \right. \\ &\quad \left. \times \zeta(4k+2) - 4 \sum_{j=1}^k \zeta(4j-1) \zeta(4k-4j+3) \right\}. \end{aligned}$$

# $q$ -Calculus

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $q \neq 1$ .

**Definition.** The  $q$ -difference operator  $d_q$  is defined by

$$(d_q f)(x) = f(qx) - f(x).$$

**Example.** For the identity map  $\iota$ , since  $\iota(x) = x$ , we have

$$(d_q \iota)(x) = qx - x = (q - 1)x.$$

**Definition.** The  $q$ -derivative operator  $D_q$  is defined by

$$(D_q f)(x) = \frac{(d_q f)(x)}{(d_q \iota)(x)} = \frac{f(qx) - f(x)}{(q - 1)x}, \quad x \neq 0.$$

**Notation.** If  $y = f(x)$ , then

$$\frac{d_q y}{d_q x} = (D_q f)(x).$$

# The $q$ -Derivative

Recall that if  $y = f(x)$ , then the  $q$ -derivative is

$$\frac{d_q y}{d_q x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}.$$

If  $f$  is differentiable at  $x \neq 0$ , then we recover the ordinary derivative  $f'(x)$  from the  $q$ -derivative in the limit as  $q \rightarrow 1$ .

To see this, substitute  $q = 1 + h/x$ . Then

$$\lim_{q \rightarrow 1} \frac{f(qx) - f(x)}{(q - 1)x} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

# The $q$ -Product Rule

Recall that if  $y = f(x)$ , then the  $q$ -derivative is

$$\frac{d_q y}{d_q x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}.$$

Therefore, the  $q$ -derivative of the product of two functions  $f, g$  is given by

$$\begin{aligned}(D_q f g)(x) &= \frac{f(qx)g(qx) - f(x)g(x)}{(q - 1)x} \\ &= \frac{f(qx)[g(qx) - g(x)] + [f(qx) - f(x)]g(x)}{(q - 1)x} \\ &= f(qx)(D_q g)(x) + g(x)(D_q f)(x).\end{aligned}$$

# The $q$ -Integral

**J. Thomae (1869):** Let  $0 < q < 1$ . Then

$$\int_0^1 f(t) d_q t := (1 - q) \sum_{n=0}^{\infty} q^n f(q^n).$$

**F. H. Jackson (1910):**

$$\int_a^b f(t) d_q t := \int_0^b f(t) d_q t - \int_0^a f(t) d_q t,$$

$$\int_0^b f(t) d_q t := (1 - q) \sum_{n=0}^{\infty} b q^n f(b q^n).$$

If  $f : [0..b] \rightarrow \mathbf{R}$  is continuous then

$$\lim_{q \rightarrow 1} \int_0^b f(t) d_q t = \int_0^b f(t) dt.$$

# Fundamental Theorem of $q$ -Calculus

Suppose  $f : (0, b] \rightarrow \mathbf{R}$  and  $0 < x \leq b$ .

The Jackson  $q$ -integral of  $f$  is defined by

$$\int_0^x f(t) d_q t := (1 - q) \sum_{j=0}^{\infty} x q^j f(x q^j).$$

If there exists  $0 \leq \alpha < 1$  such that  $|f(t)t^\alpha|$  is bounded on  $(0, b]$ , then the integral converges to a function  $F(x)$  on  $(0, b]$ .

Additionally,  $F$  is a  $q$ -antiderivative of  $f$ :

$$D_q F(x) := \frac{F(qx) - F(x)}{(q - 1)x} = f(x), \quad 0 < x \leq b.$$

Note that

$$\lim_{q \rightarrow 1} D_q F(x) = F'(x), \quad \text{and} \quad \lim_{q \rightarrow 1} \int_0^x f(t) d_q t = \int_0^x f(t) dt.$$

# The $q$ -analog of $\alpha$

Recall that if  $y = f(x)$ , then

$$\frac{d_q y}{d_q x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

**Example.** Let  $\alpha \in \mathbf{R}$  and  $y = x^\alpha$ . Then

$$\frac{d_q y}{d_q x} = \frac{(qx)^\alpha - x^\alpha}{(q-1)x} = \left( \frac{q^\alpha - 1}{q-1} \right) x^{\alpha-1},$$

$$\lim_{q \rightarrow 1} \frac{d_q y}{d_q x} = \frac{dy}{dx} = \alpha x^{\alpha-1}.$$

**Definition.** Let  $\alpha \in \mathbf{R}$  and  $q \neq 1$ . The  $q$ -analog of  $\alpha$  is

$$[\alpha]_q := \frac{q^\alpha - 1}{q-1}.$$



# Multiple $q$ -Zeta Values

M. Kaneko (2003) investigated analytic properties of the Riemann  $q$ -zeta function

$$\zeta[s] := \sum_{k=1}^{\infty} \frac{q^{tk}}{[k]_q^s}, \quad t = s - 1.$$

This suggested the definition

$$\zeta[s_1, \dots, s_m] := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Alternatively, we could let

$$Z[s_1, \dots, s_m] := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

If  $\vec{s} = (s_1, \dots, s_m)$ , then

$$\zeta[\vec{s}; q] = q^{|\vec{s}|} Z[\vec{s}; 1/q], \quad |\vec{s}| := \sum_{j=1}^m s_j.$$

# Period-1 Sums Reduce

**Theorem 2** *If  $n$  is a positive integer and  $s > 1$ , then*

$$n\zeta[\{s\}^n] = \sum_{k=1}^n (-1)^{k+1} \zeta[\{s\}^{n-k}] \sum_{j=0}^{k-1} \binom{k-1}{j} (1-q)^j \zeta[ks-j].$$

**Example 1** *With  $n = 2$ , we get*

$$2\zeta[s, s] = \zeta[s]\zeta[s] - \left( \zeta[2s] + (1-q)\zeta[2s-1] \right).$$

**Corollary 1** *If  $n$  is a positive integer and  $s > 1$ , then*

$$n\zeta(\{s\}^n) = \sum_{k=1}^n (-1)^{k+1} \zeta(\{s\}^{n-k}) \zeta(ks).$$

# Sums Over Permutations

Let  $\mathfrak{S}_n$  denote the group of  $n!$  permutations of  $\langle n \rangle = \{1, 2, \dots, n\}$ .

**Theorem 3** *Let  $n$  be a positive integer, and let  $s_j > 1$  for  $1 \leq j \leq n$ . Then*

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta \left[ \mathbf{Cat}_{j=1}^n s_{\sigma(j)} \right] = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{k=1}^{|\mathcal{P}|} (|P_k| - 1)! \\ \times \sum_{\nu_k=0}^{|P_k|-1} \binom{|P_k|-1}{\nu_k} (1-q)^{\nu_k} \zeta[p_k - \nu_k],$$

where the outer sum on the right is over all unordered set partitions  $\mathcal{P} = \{P_1, \dots, P_m\}$  of  $\langle n \rangle$ ,  $1 \leq m = |\mathcal{P}| \leq n$ , and  $p_k = \sum_{j \in P_k} s_j$ .

**Corollary 2 (M. Hoffman)**

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta \left( \mathbf{Cat}_{j=1}^n s_{\sigma(j)} \right) = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{P \in \mathcal{P}} (|P| - 1)! \zeta \left( \sum_{j \in P} s_j \right).$$

# Parity Reduction

**Theorem 4** *Let  $m \in \mathbf{Z}^+$  and let  $s_1, \dots, s_m$  be real numbers with  $s_1 > 1$ ,  $s_m > 1$ , and  $s_j \geq 1$  for  $1 < j < m$ . Then*

$$\zeta \left[ \mathbf{Cat}_{k=1}^m s_k \right] + (-1)^m \zeta \left[ \mathbf{Cat}_{k=1}^m s_{m-k+1} \right]$$

*can be expressed as a  $\mathbf{Z}[q]$ -linear combination of multiple  $q$ -zeta values of depth less than  $m$ .*

In other words, the coefficients in the linear combination are polynomials in  $q$  with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

# A Double Generating Function

## Theorem 5

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} \zeta[m+2, \{1\}^n] \\ &= 1 - \exp \left\{ \sum_{k=2}^{\infty} \left\{ u^k + v^k - (u + v + (1-q)uv)^k \right\} \right. \\ & \quad \left. \times \frac{1}{k} \sum_{j=2}^k (q-1)^{k-j} \zeta[j] \right\}. \end{aligned}$$

**Corollary 3** *If  $0 \leq m, n \in \mathbf{Z}$ , then*

$$\zeta[m+2, \{1\}^n] = \zeta[n+2, \{1\}^m].$$

**Corollary 4 ( $q$ -Euler convolution)** *Let  $0 \leq m \in \mathbf{Z}$ . Then*

$$\begin{aligned} 2\zeta[m+2, 1] &= (m+2)\zeta[m+3] + (1-q)m\zeta[m+2] \\ & \quad - \sum_{k=2}^{m+1} \zeta[m+3-k] \zeta[k]. \end{aligned}$$

The proof of Theorem 5 makes essential use of the basic hypergeometric function

$${}_2\phi_1 \left[ \begin{matrix} q^a, q^b \\ q^c \end{matrix} \middle| x \right] = 1 + \sum_{n=1}^{\infty} x^n \prod_{k=0}^{n-1} \frac{(1 - q^{a+k})(1 - q^{b+k})}{(1 - q^{c+k})(1 - q^{1+k})}, \quad |x| < 1.$$

Routine series manipulations reveal that

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} [x]_q^{m+1} [y]_q^{n+1} \zeta[m+2, \{1\}^n] \\ = 1 - {}_2\phi_1 \left[ \begin{matrix} q^{-y}, q^x \\ q^{1+x} \end{matrix} \middle| q^{1+y} \right]. \end{aligned}$$

Heine's  $q$ -analog

$${}_2\phi_1 \left[ \begin{matrix} q^a, q^b \\ q^c \end{matrix} \middle| q^{c-a-b} \right] = \frac{\Gamma_q(c)\Gamma_q(c-a-b)}{\Gamma_q(c-a)\Gamma_q(c-b)}$$

of Gauss's  ${}_2F_1$  summation formula then gives

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} [x]_q^{m+1} [y]_q^{n+1} \zeta[m+2, \{1\}^n] \\ = 1 - \frac{\Gamma_q(1+x)\Gamma_q(1+y)}{\Gamma_q(1+x+y)}, \end{aligned}$$

where for  $0 < q < 1$ ,

$$\Gamma_q(1+x) := (1-q)^{-x} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{n+x})},$$

is the  $q$ -analog of Euler's gamma function.

If  $\vec{s} = (s_1, \dots, s_m)$  define

$$\begin{aligned} \text{weight}(\vec{s}) &:= |\vec{s}| = \sum_{j=1}^m s_j, \\ \text{depth}(\vec{s}) &:= m, \\ \text{height}(\vec{s}) &:= \#\{j : s_j \geq 2\}. \end{aligned}$$

### Theorem 6 (J. Okuda & Y. Takeyama)

$$\begin{aligned} &1 + (w - uv) \sum_{s, m, h \geq 0} u^{s-m-h} v^{m-h} w^{h-1} \sum_{\substack{\text{weight}(\vec{s})=s \\ \text{depth}(\vec{s})=m \\ \text{height}(\vec{s})=h}} \zeta[\vec{s}] \\ &= \exp \left\{ \sum_{k=2}^{\infty} (u^k + v^k - \alpha^k - \beta^k) \frac{1}{k} \sum_{j=2}^k (q-1)^{k-j} \zeta[j] \right\}, \end{aligned}$$

where  $\alpha$  and  $\beta$  satisfy the equations

$$\alpha + \beta = u + v + (q-1)(w - uv), \quad \alpha\beta = w.$$

Theorem 5 is case  $w = 0$  of Theorem 6.



# The Simplex Integral

M. Kontsevich: If  $s_1, \dots, s_m \in \mathbf{Z}^+$ , then

$$\zeta(s_1, \dots, s_m) = \int \prod_{k=1}^m \left( \prod_{r=1}^{s_k-1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \dots > t_{s_1}^{(1)} > \dots > t_1^{(m)} > \dots > t_{s_m}^{(m)} > 0,$$

and is abbreviated (D. Broadhurst) by

$$\int_0^1 \prod_{k=1}^m A^{s_k-1} B, \quad A = \frac{dt}{t}, \quad B = \frac{dt}{1-t}.$$

# Example

$$\begin{aligned}
 \zeta(2, 1) &= \sum_{n>m>0} n^{-2}m^{-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k+j)^{-2}k^{-1} \\
 &= \sum_{k=1}^{\infty} k^{-1} \sum_{j=1}^{\infty} (k+j)^{-1} \int_0^1 t^{k+j-1} dt \\
 &= \sum_{k=1}^{\infty} k^{-1} \int_0^1 t^{-1} \sum_{j=1}^{\infty} \int_0^t u^{k+j-1} du dt \\
 &= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} k^{-1} u^k du dt \\
 &= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} \int_0^u v^{k-1} dv du dt \\
 &= \int_{1>t>u>v>0} \frac{dt}{t} \cdot \frac{du}{1-u} \cdot \frac{dv}{1-v} \\
 &= \int_0^1 AB^2.
 \end{aligned}$$

# The Jackson $q$ -Integral

Suppose  $f : (0, b] \rightarrow \mathbf{R}$  and  $0 < x \leq b$ .

Recall the Jackson  $q$ -integral of  $f$  on the subinterval  $(0, x]$  is

$$\int_0^x f(t) d_q t := (1 - q) \sum_{j=0}^{\infty} x q^j f(x q^j),$$

and if there exists  $0 \leq \alpha < 1$  such that  $|f(t)t^\alpha|$  is bounded on  $(0, b]$ , then the integral converges to a function  $F(x)$  on  $(0, b]$ .

Additionally (fundamental theorem of  $q$ -calculus),  $F$  is a  $q$ -antiderivative of  $f$ :

$$D_q F(x) := \frac{F(qx) - F(x)}{(q - 1)x} = f(x), \quad 0 < x \leq b.$$

# The Jackson Simplex Integral

Let  $s_1, \dots, s_m$  are positive integers. Recall:

$$\zeta(s_1, \dots, s_m) = \int \prod_{k=1}^m \left( \prod_{r=1}^{s_k-1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \dots > t_{s_1}^{(1)} > \dots > t_1^{(m)} > \dots > t_{s_m}^{(m)} > 0.$$

## Theorem 7

$$\zeta[s_1, \dots, s_m] = \int \prod_{k=1}^m \left( \prod_{r=1}^{s_k-1} \frac{d_q t_r^{(k)}}{t_r^{(k)}} \right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}^{(k)}},$$

where

$$y_k := \prod_{j=1}^k q^{1-s_j},$$

and the integral is over the same simplex as above.

# Duality

Let  $a_i, b_i \in \mathbf{Z}^+$  and  $k = \sum_{i=1}^n (a_i + b_i)$ . Then

$$\begin{aligned} \zeta(a_1 + 1, \{1\}^{b_1-1}, \dots, a_n + 1, \{1\}^{b_n-1}) &= \int_0^1 \prod_{i=1}^n A^{a_i} B^{b_i} \\ &= \int_{1 > t_1 > \dots > t_k > 0} \prod_{j=1}^k f_j(t_j) dt_j \\ &= \int_{1 > u_k > \dots > u_1 > 0} \prod_{j=1}^k f_j(u_j) du_j, \quad u_j = 1 - t_j \\ &= \int_0^1 \prod_{i=n}^1 A^{b_i} B^{a_i} = \zeta(b_n + 1, \{1\}^{a_n-1}, \dots, b_1 + 1, \{1\}^{a_1-1}). \end{aligned}$$

# Generalized Duality

**Definition 1** Let  $n$  and  $s_1, \dots, s_n$  be positive integers with  $s_1 > 1$ . Let  $m$  be a non-negative integer. Define

$$S(s_1, \dots, s_n; m) := \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = m}} \zeta(s_1 + c_1, \dots, s_n + c_n).$$

For positive integers  $a_i$  and  $b_i$ , define the dual argument lists

$$\begin{aligned} \vec{s} &= \mathbf{Cat}_{i=1}^n \{a_i + 1, \{1\}^{b_i - 1}\}, \\ \vec{s}' &= \mathbf{Cat}_{i=n}^1 \{b_i + 1, \{1\}^{a_i - 1}\}. \end{aligned}$$

**Theorem 8 (Y. Ohno)** For any pair of dual argument lists  $\vec{s}$ ,  $\vec{s}'$  and any non-negative integer  $m$ , we have the equality

$$S(\vec{s}; m) = S(\vec{s}'; m).$$

# Generalized $q$ -Duality

**Definition 2** Let  $n$  and  $s_1, \dots, s_n$  be positive integers with  $s_1 > 1$ . Let  $m$  be a non-negative integer. Define

$$S[s_1, \dots, s_n; m] := \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = m}} \zeta[s_1 + c_1, \dots, s_n + c_n].$$

For positive integers  $a_i$  and  $b_i$ , define the dual argument lists

$$\begin{aligned} \vec{s} &= \mathbf{Cat}_{i=1}^n \{a_i + 1, \{1\}^{b_i - 1}\} \\ \vec{s}' &= \mathbf{Cat}_{i=n}^1 \{b_i + 1, \{1\}^{a_i - 1}\}. \end{aligned}$$

**Theorem 9** For any pair of dual argument lists  $\vec{s}$ ,  $\vec{s}'$  and any non-negative integer  $m$ , we have

$$S[\vec{s}; m] = S[\vec{s}'; m].$$

## $q$ -Duality

**Corollary 5** *If  $\vec{s}, \vec{s}'$  are dual argument lists, then*

$$\zeta[\vec{s}] = \zeta[\vec{s}'].$$

*In other words, if  $a_i, b_i \in \mathbf{Z}^+$  ( $1 \leq i \leq n$ ), then*

$$\zeta\left[\mathbf{Cat}_{i=1}^n \{a_i + 1, \{1\}^{b_i-1}\}\right] = \zeta\left[\mathbf{Cat}_{i=n}^1 \{b_i + 1, \{1\}^{a_i-1}\}\right].$$

**Proof.** Put  $m = 0$  in Theorem 9 (generalized  $q$ -duality).

□



## $q$ -Sum Formula

**Definition 3** Let  $t_1, \dots, t_n$  be positive integers.

$$\zeta^*[t_1, \dots, t_n] := \zeta\left[t_1 + 1, \mathbf{Cat}_{j=2}^n t_j\right].$$

**Corollary 6 ( $q$ -Sum Formula)** For any integers  $0 < k \leq n$ , we have

$$\sum_{t_1+t_2+\dots+t_n=k} \zeta^*[t_1, t_2, \dots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers  $t_1, \dots, t_n$  with sum equal to  $k$ .

**Proof.** If we take the dual argument lists in the form  $\vec{s} = (n + 1)$  and  $\vec{s}' = (2, \{1\}^{n-1})$  and put  $m = k - n$ , then Theorem 9 (generalized  $q$ -duality) states that

$$\begin{aligned} \zeta[k + 1] &= \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = k - n}} \zeta\left[2 + c_2, \mathbf{Cat}_{j=2}^n \{1 + c_j\}\right] \\ &= \sum_{\substack{t_1, \dots, t_n \geq 1 \\ t_1 + \dots + t_n = k}} \zeta\left[t_1 + 1, \mathbf{Cat}_{j=2}^n t_j\right]. \end{aligned}$$

□

# $q$ -Cyclic Sum Formula

**Definition 4** Let  $s_j \in \mathbf{Z}^+$  for  $1 \leq j \leq n$  and put  $\vec{s} = (s_1, \dots, s_n)$ . Let  $\sigma$  denote the  $n$ -cycle  $(1\ 2 \cdots n)$ , and let

$$\mathcal{C}(\vec{s}) := \{(s_{\sigma^j(1)}, \dots, s_{\sigma^j(n)}) : 1 \leq j \leq n\}$$

denote the set of cyclic permutations of  $\vec{s}$ .

Recall the definition

$$\zeta^*[s_1, \dots, s_n] := \zeta[s_1 + 1, s_2, \dots, s_n].$$

**Theorem 10** Let  $\vec{s}$  and  $\vec{s}'$  be dual argument lists. Then

$$\sum_{\vec{t} \in \mathcal{C}(\vec{s})} \zeta^*[\vec{t}] = \sum_{\vec{t} \in \mathcal{C}(\vec{s}')} \zeta^*[\vec{t}].$$

# Reformulation of $q$ -Duality

Let  $\mathfrak{h} = \mathbf{Q}\langle x, y \rangle$  denote the non-commutative polynomial algebra over the rational numbers in two indeterminates  $x$  and  $y$ .

Let  $\mathfrak{h}^0$  denote the subalgebra  $\mathbf{Q}1 \oplus x\mathfrak{h}y$ . The  $\mathbf{Q}$ -linear map  $\hat{\zeta}$  is defined on  $\mathfrak{h}^0$  by

$$\hat{\zeta}[1] := \zeta[1] = 1$$

and

$$\hat{\zeta}\left[\prod_{i=1}^s x^{a_i} y^{b_i}\right] = \zeta\left[\mathbf{Cat}_{i=1}^s \left\{a_i + 1, \{1\}^{b_i-1}\right\}\right],$$

for positive integers  $a_i, b_i$  ( $1 \leq i \leq s$ ).

Let  $\tau$  be the anti-automorphism of  $\mathfrak{h}$  that switches  $x$  and  $y$ .

Then  $q$ -duality simply says that

$$\hat{\zeta}[\tau w] = \hat{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.$$

# Derivations

**Definition 5 (K. Ihara & M. Kaneko)** Define a derivation on  $\mathfrak{h}$  for each positive integer  $n$  by

$$\partial_n(x) = x(x + y)^{n-1}, \quad \partial_n(y) = -x(x + y)^{n-1}y.$$

**Theorem 11 (Ihara & Kaneko)** For all positive integers  $n$  and words  $w \in \mathfrak{h}^0$ ,  $\widehat{\zeta}(\partial_n(w)) = 0$ .

**Theorem 12 ( $q$ -Analog)** For all positive integers  $n$  and words  $w \in \mathfrak{h}^0$ ,  $\widehat{\zeta}[\partial_n(w)] = 0$ .

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Theorem 12 is actually *equivalent* to generalized  $q$ -duality (Theorem 9).

# Proof of Theorem 12

**Proof.** Let  $\sigma = \exp(\Delta)$ ,  $\tilde{\sigma} = \tau\sigma\tau$ ,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n, \quad \partial = \sum_{n=1}^{\infty} \frac{\partial_n}{n} \theta^n.$$

Generalized  $q$ -duality (Theorem 9):  $\forall w \in \mathfrak{h}^0$ ,

$$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w] = \widehat{\zeta}[\tau\sigma\tau w] \iff (\sigma - \tilde{\sigma})w \in \ker \widehat{\zeta}.$$

We show that in fact,  $(\sigma - \tilde{\sigma})\mathfrak{h}^0 = \partial\mathfrak{h}^0$ .

To prove this, we require the following identity of Ihara and Kaneko.

**Proposition 13**  $\exp(\partial) = \tilde{\sigma}\sigma^{-1}$ .

To complete the proof of Theorem 12, observe that since

$$\begin{aligned}\partial &= \log(\tilde{\sigma}\sigma^{-1}) = \log(1 - (\sigma - \tilde{\sigma})\sigma^{-1}) \\ &= -(\sigma - \tilde{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n} \left( (\sigma - \tilde{\sigma})\sigma^{-1} \right)^{n-1} \sigma^{-1},\end{aligned}$$

and

$$\begin{aligned}\sigma - \tilde{\sigma} &= (1 - \tilde{\sigma}\sigma^{-1})\sigma = (1 - \exp(\partial))\sigma \\ &= -\partial \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{n!} \sigma,\end{aligned}$$

we see that

$$\partial\mathfrak{h}^0 \subseteq (\sigma - \tilde{\sigma})\mathfrak{h}^0 \quad \text{and} \quad (\sigma - \tilde{\sigma})\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0.$$

Thus for the kernel of  $\hat{\zeta}$ , we have the equivalences

$$\begin{aligned}(\sigma - \tilde{\sigma})w \in \ker \hat{\zeta} &\iff \partial w \in \ker \hat{\zeta} \\ &\iff \forall n \in \mathbf{Z}^+, \hat{\zeta}[\partial_n w] = 0.\end{aligned}$$

## $q$ -analog of Euler's Decomposition Formula

The respective  $q$ -analogs of the Riemann and double zeta functions are

$$\zeta[s] = \sum_{n>0} \frac{q^{(s-1)n}}{[n]_q^s} \quad \text{and} \quad \zeta[s, t] = \sum_{n>k>0} \frac{q^{(s-1)n} q^{(k-1)t}}{[n]_q^s [k]_q^t}.$$

We also need the sum

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s].$$



**Theorem 14** *If  $s - 1$  and  $t - 1$  are positive integers, then*

$$\begin{aligned} \zeta[s]\zeta[t] &= \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} (1-q)^b \zeta[t+a, s-a-b] \\ &+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b \zeta[s+a, t-a-b] \\ &- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \cdot \frac{(1-q)^j}{(j-1)!} \varphi[s+t-j]. \end{aligned}$$

Observe that the limiting case  $q = 1$  of Theorem 14 reduces to Euler's decomposition formula

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a).$$

# A Differential Identity

The proof of Theorem 14 relies on the following identity.

**Lemma 15** *Let  $s$  and  $t$  be positive integers, and let  $x$  and  $y$  be non-zero real numbers. Then for all real  $q$  such that  $x + y + (q - 1)xy \neq 0$ ,*

$$\begin{aligned}
 & \frac{1}{x^s y^t} \\
 &= \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} \frac{(1-q)^b (1+(q-1)y)^a (1+(q-1)x)^{t-1-b}}{x^{s-a-b} (x+y+(q-1)xy)^{t+a}} \\
 &+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} \frac{(1-q)^b (1+(q-1)x)^a (1+(q-1)y)^{s-1-b}}{y^{t-a-b} (x+y+(q-1)xy)^{s+a}} \\
 &- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \cdot \frac{(1-q)^j}{(j-1)!} \cdot \frac{(1+(q-1)y)^{s-j} (1+(q-1)x)^{t-j}}{(x+y+(q-1)xy)^{s+t-j}}.
 \end{aligned}$$

# Proof of the Differential Identity

Apply the partial differential operator

$$\frac{1}{(s-1)!} \left( -\frac{\partial}{\partial x} \right)^{s-1} \frac{1}{(t-1)!} \left( -\frac{\partial}{\partial y} \right)^{t-1}$$

to both sides of the identity

$$\frac{1}{xy} = \frac{1}{x+y+(q-1)xy} \left( \frac{1}{x} + \frac{1}{y} + q - 1 \right).$$

□

Observe that when  $q = 1$ , Lemma 15 reduces to the identity

$$\frac{1}{x^s y^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a} (x+y)^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{(x+y)^{s+a} y^{t-a}},$$

from which Euler's decomposition formula for  $\zeta(s)\zeta(t)$  follows immediately on summing over all positive integers  $x$  and  $y$ .

The more general  $q$ -decomposition formula for  $\zeta[s]\zeta[t]$  is obtained by first putting  $x = [u]_q$  and  $y = [v]_q$  in Lemma 15 and summing over all positive integers  $u$  and  $v$ .

Note that if  $x = [u]_q$  and  $y = [v]_q$ , then  $1+(q-1)x = q^u$ ,  $1+(q-1)y = q^v$  and

$$[u+v]_q = x + y + (q-1)xy = [u]_q + [v]_q + (q-1)[u]_q[v]_q.$$

# $q$ -analog of Euler's reduction formula

Introduce additional  $q$ -analogs of  $\zeta(s, t)$  by defining

$$\zeta_1[s, t] := (-1)^t \sum_{u > v > 0}^{\infty} \frac{q^{(s-1)u + (t-1)(-v)}}{[u]_q^s [-v]_q^t} = \sum_{u > v > 0}^{\infty} \frac{q^{(s-1)u + v}}{[u]_q^s [v]_q^t}$$

and

$$\zeta_2[s, t] := (-1)^s \sum_{u > v > 0}^{\infty} \frac{q^{(s-1)(-u) + (t-1)v}}{[-u]_q^s [v]_q^t} = \sum_{u > v > 0}^{\infty} \frac{q^{u + (t-1)v}}{[u]_q^s [v]_q^t}.$$

Let

$$\zeta_-[s] := \sum_{n=1}^{\infty} \frac{q^{(s-1)(-n)}}{[-n]_q^s} = (-1)^s \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s}$$

and for convenience, put

$$\zeta_{\pm}[s] := \zeta[s] + \zeta_-[s] = \sum_{0 \neq n \in \mathbf{Z}} \frac{q^{(s-1)n}}{[n]_q^s}.$$

As defined previously, let

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s].$$

We also employ the notation

$$\binom{z}{a, b} := \binom{z}{a} \binom{z-a}{b} = \binom{z}{b} \binom{z-b}{a} = \binom{z}{a+b} \frac{(a+b)!}{a!b!}$$

for the trinomial coefficient, in which  $a, b$  are nonnegative integers, and which reduces to  $z!/a!b!(z-a-b)!$  if  $z$  is an integer not less than  $a+b$ .

**Theorem 16 ( $q$ -analog of Euler's double zeta reduction)** *Let  $s > 1$  and  $t > 1$  be integers, and let  $0 < q < 1$ . Then*

$$\begin{aligned}
& (-1)^t \zeta_1[s, t] - (-1)^s \zeta_2[s, t] \\
&= \sum_{a=0}^{s-2} \sum_{b=0}^{s-2-a} \binom{a+t-1}{a, b} (1-q)^b \left( \zeta_{\pm}[s-a-b] \zeta[a+t] \right. \\
&\quad \left. - \zeta[s+t-b] - (1-q) \zeta[s+t-b-1] \right) \\
&+ \sum_{a=0}^{t-2} \sum_{b=0}^{t-2-a} \binom{a+s-1}{a, b} (1-q)^b \left( \zeta_{\pm}[t-a-b] \zeta[a+s] \right. \\
&\quad \left. - \zeta[s+t-b] - (1-q) \zeta[s+t-b-1] \right) \\
&- \sum_{j=1}^{\min(s,t)} \binom{s+t-j-1}{s-j, t-j} (1-q)^{j-1} \left( 2\zeta[s+t-j+1] - (1-q)\varphi[s+t-j] \right) \\
&- \zeta_{\pm}[s] \zeta[t] + (-1)^s \sum_{k=0}^{s-1} \binom{s-1}{k} (1-q)^k \zeta[s+t-k].
\end{aligned}$$

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