Multiple Mordell-Tornheim series and q-analogs

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The Tornheim double zeta function is defined by the double series

$$T(s,t,u) := \sum_{m,n=1}^{\infty} \frac{1}{m^s n^t (m+n)^u},$$

which converges absolutely for complex numbers s, t and u such that $\Re(s+u) > 1$, $\Re(t+u) > 1$ and $\Re(s+t+u) > 2$.

It is a generalization of the Euler double zeta function

$$\zeta(u,s) := \sum_{m,n=1}^{\infty} \frac{1}{m^s (m+n)^u} = T(s,0,u)$$

defined for complex s and u such that $\Re(u) > 1$ and $\Re(s+u) > 2$.

The double zeta function can be viewed as a bivariate extension of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1.$$

Tornheim (1950): If a, b and c are positive integers with $a+b+c$ odd, then $T(a,b,c) \in \mathbb{Q}[\{\zeta(2k)\zeta(a+b+c-2k) : k \in \mathbb{Z}, 0 \le k \le (a+b+c-3)/2\}]$.

Huard, Williams, Zhang (1996): If a, b and c are positive integers with $a+b+c$ odd, then

$$T(a,b,c) = (-1)^a \sum_{k=0}^{\max(a,c)/2} \left\{ \binom{a+c-2k-1}{a-1} + \binom{a+c-2k-1}{c-1} \right\} \times \zeta(2k)\zeta(a+b+c-2k)$$
$$+ (-1)^b \sum_{k=0}^{\max(b,c)/2} \left\{ \binom{b+c-2k-1}{b-1} + \binom{b+c-2k-1}{c-1} \right\} \times \zeta(2k)\zeta(a+b+c-2k).$$

When $a+b+c$ is even, the situation is more complicated.

Subbarao and Sitaramachandrarao (1985): If $a, b, c \in \mathbb{Z}^+$, then

$$T(2a,2b,2c) + T(2b,2c,2a) + T(2c,2a,2b)$$

$$= \frac{4}{(2a)!(2b)!} \sum_{k=0}^{\max(a,b)} \left\{ a\binom{2b}{2k} + b\binom{2a}{2k} \right\} (2a+2b-2k-1)!\,(2k)! \times \zeta(2k)\,\zeta(2a+2b+2c-2k).$$

$$\Rightarrow T(2a,2a,2a) = \frac{4}{3} \sum_{k=0}^{a} \binom{4a-2k-1}{2a-1} \zeta(2k)\zeta(6a-2k).$$

Tsumura (2006) evaluated $T(a,b,s) + (-1)^b T(b,s,a) + (-1)^a T(s,a,b)$ for positive integers a, b and complex s.

Nakamura (2006) gave a simpler evaluation of the same quantity.

Theorem 1 (Nakamura) *If a and b are positive integers and s is a complex number such that* $\min(a,b)+\Re(s)>1$, *then*

$$T(a,b,s)+(-1)^{b}T(b,s,a)+(-1)^{a}T(s,a,b)$$

$$=\frac{2}{a!\,b!}\sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}(a+b-2k-1)!\,(2k)!$$
$$\times\zeta(2k)\zeta(a+b+s-2k).$$

Nakamura and Subbarao/Sitaramachandrarao both base their approaches on an identity for the product of two Bernoulli polynomials.

A much simpler approach can be based instead on the trivial identity

$$\frac{1}{xy}=\left(\frac{1}{x}+\frac{1}{y}\right)\frac{1}{x+y}.$$

For convenience, we restate Nakamura's result with the right hand side further simplified:

Theorem 2 (Nakamura) *If a and b are positive integers and s is a complex number such that* $\min(a,b)+\Re(s)>1$, *then*

$$T(a,b,s)+(-1)^{b}T(b,s,a)+(-1)^{a}T(s,a,b)$$

$$=2\sum_{k=0}^{\max(a,b)/2}\left\{\binom{a+b-2k-1}{a-1}+\binom{a+b-2k-1}{b-1}\right\}$$
$$\times\zeta(2k)\zeta(a+b+s-2k).$$

Proof of Theorem 2

For integers a,b and complex s with $\min(a,b)+\Re(s)>1$ and $a+b>1$, let

$$S(a,b,s):=T(a,b,s)+(-1)^b T(b,s,a)+(-1)^a T(s,a,b),$$

and define

$$S(0,1,s)=S(1,0,s):=-\zeta(s+1) \quad \text{for} \quad \Re(s)>0.$$

Note that S is symmetric with respect to the interchange of its first two arguments, i.e. $S(a,b,s)=S(b,a,s)$ whenever either quantity is defined.

Lemma 3 *If a and b are positive integers and s is a complex number with $\min(a,b)+\Re(s)>1$, then*

$$S(a,b,s)=\sum_{j=0}^{a-1}\binom{j+b-1}{b-1}S(a-j,0,b+s+j) + \sum_{j=0}^{b-1}\binom{j+a-1}{a-1}S(0,b-j,a+s+j).$$

Proof. Let a and b be positive integers. Apply the partial differential operator

$$\frac{1}{(a-1)!}\left(-\frac{\partial}{\partial x}\right)^{a-1}\frac{1}{(b-1)!}\left(-\frac{\partial}{\partial y}\right)^{b-1}$$

to both sides of the identity

$$\frac{1}{xy}=\left(\frac{1}{x}+\frac{1}{y}\right)\frac{1}{x+y}.$$

This yields

$$\frac{1}{x^a y^b} = \sum_{j=0}^{a-1} \binom{j+b-1}{b-1} \frac{1}{x^{a-j}(x+y)^{b+j}} + \sum_{j=0}^{b-1} \binom{j+a-1}{a-1} \frac{1}{y^{b-j}(x+y)^{a+j}}. \qquad (1)$$

Now exchange a and b in (1), add the two resulting equations, replace x by $x-y$ and then replace y by $-y$ to obtain

$$\frac{(-1)^a}{y^a(x+y)^b} + \frac{(-1)^b}{y^b(x+y)^a} = \sum_{j=0}^{a-1} \binom{j+b-1}{b-1} \left[\frac{(-1)^{a-j}}{x^{b+j} y^{a-j}} + \frac{1}{x^{b+j}(x+y)^{a-j}} \right] + \sum_{j=0}^{b-1} \binom{j+a-1}{a-1} \left[\frac{1}{x^{a+j}(x+y)^{b-j}} + \frac{(-1)^{b-j}}{x^{a+j} y^{b-j}} \right]. \qquad (2)$$

Now multiply (1) by $1/(x+y)^s$, multiply (2) by $1/x^s$ and add the resulting two equations.

If we separate the final terms in each sum, then

$$\frac{1}{x^a y^b (x+y)^s} + \frac{(-1)^a}{x^s y^a (x+y)^b} + \frac{(-1)^b}{x^s y^b (x+y)^a}$$

$$= \sum_{j=0}^{a-2} \binom{j+b-1}{b-1} \left[\frac{1}{x^{a-j}(x+y)^{b+s+j}} + \frac{(-1)^{a-j}}{x^{b+s+j} y^{a-j}} + \frac{1}{x^{b+s+j}(x+y)^{a-j}} \right]$$

$$+ \sum_{j=0}^{b-2} \binom{j+a-1}{a-1} \left[\frac{1}{y^{b-j}(x+y)^{a+s+j}} + \frac{(-1)^{b-j}}{x^{a+s+j} y^{b-j}} + \frac{1}{x^{a+s+j}(x+y)^{b-j}} \right]$$

$$+ \binom{a+b-2}{b-1} \left[\frac{1}{x(x+y)^{a+b+s-1}} - \frac{1}{x^{a+b+s-1} y} + \frac{1}{x^{a+b+s-1}(x+y)} \right]$$

$$+ \binom{a+b-2}{a-1} \left[\frac{1}{y(x+y)^{a+b+s-1}} - \frac{1}{x^{a+b+s-1} y} + \frac{1}{x^{a+b+s-1}(x+y)} \right]. \qquad (3)$$

Summing (3) over all positive integers x and y and noting that

$$\sum_{x,y=1}^{\infty} \frac{1}{y(x+y)^{a+b+s-1}} - \sum_{x,y=1}^{\infty} \frac{1}{x^{a+b+s-1}}\left(\frac{1}{y} - \frac{1}{x+y}\right)$$

$$= T(1,0,a+b+s-1) - \sum_{x=1}^{\infty} \frac{1}{x^{a+b+s-1}} \sum_{y=1}^{x} \frac{1}{y}$$

$$= \zeta(a+b+s-1,1) - \zeta(a+b+s) - \zeta(a+b+s-1,1)$$

$$= S(1,0,a+b+s-1)$$

completes the proof of Lemma 3. $\square$

Now note that if $a > 1$ and $\Re(s) > 1$, then the stuffle relation

$$T(s,a,0) = \zeta(a)\zeta(s) = T(a,0,s) + T(s,0,a) + \zeta(a+s)$$

implies that

$$\begin{aligned} S(0,a,s) = S(a,0,s) &= T(a,0,s) + T(s,0,a) + (-1)^a T(s,a,0) \\ &= [1 + (-1)^a]\zeta(a)\zeta(s) - \zeta(a+s). \end{aligned}$$

We substitute this and the definition $S(0,1,s) = S(1,0,s) = -\zeta(s+1)$ in the right hand side of Lemma 3 (restated below):

$$\begin{aligned} S(a,b,s) = &\sum_{j=1}^{a} \binom{a+b-j-1}{b-1} S(j,0,a+b+s-j) \\ &+ \sum_{j=1}^{b} \binom{a+b-j-1}{a-1} S(0,j,a+b+s-j), \end{aligned}$$

obtaining

$$S(a,b,s) = 2\sum_{k=1}^{\max(a,b)/2}\left\{\binom{a+b-2k-1}{a-1}+\binom{a+b-2k-1}{b-1}\right\}$$
$$\times\zeta(2k)\zeta(a+b+s-2k)$$
$$-\left\{\sum_{j=1}^{a}\binom{a+b-j-1}{b-1}+\sum_{j=1}^{b}\binom{a+b-j-1}{a-1}\right\}\zeta(a+b+s).$$

Now use the facts that

$$\sum_{j=1}^{a}\binom{a+b-j-1}{b-1}=\binom{a+b-1}{a-1},\quad \sum_{j=1}^{b}\binom{a+b-j-1}{a-1}=\binom{a+b-1}{b-1},$$

and $\zeta(0) = -1/2$ to write

$$S(a,b,s) = 2\sum_{k=0}^{\max(a,b)/2}\left\{\binom{a+b-2k-1}{a-1}+\binom{a+b-2k-1}{b-1}\right\}$$
$$\times\zeta(2k)\zeta(a+b+s-2k),$$

as claimed. ∎

Corollary 1 (Huard, Williams, Zhang) *If $a,b,c \in \mathbb{Z}^+$ and $a+b+c$ is odd, then*

$$T(a,b,c) = (-1)^a\sum_{k=0}^{\max(a,c)/2}\left\{\binom{a+c-2k-1}{a-1}+\binom{a+c-2k-1}{c-1}\right\}$$
$$\times\zeta(2k)\zeta(a+b+c-2k)$$
$$+(-1)^b\sum_{k=0}^{\max(b,c)/2}\left\{\binom{b+c-2k-1}{b-1}+\binom{b+c-2k-1}{c-1}\right\}$$
$$\times\zeta(2k)\zeta(a+b+c-2k).$$

Proof. Following Nakamura, let

$$N(a,b,c) = \frac{1}{2}S(a,b,c) = \sum_{k=0}^{\max(a,b)/2}\left\{\binom{a+b-2k-1}{a-1}+\binom{a+b-2k-1}{b-1}\right\}$$
$$\times\zeta(2k)\zeta(a+b+c-2k)$$

It suffices to show that $T(a,b,c) = (-1)^a N(c,a,b) + (-1)^b N(b,c,a)$.

But if $a,b,c \in \mathbb{Z}^+$, then Nakamura's theorem says that

$$T(a,b,c) + (-1)^b T(b,c,a) + (-1)^a T(c,a,b) = 2N(a,b,c).$$

Cycling the variables yields

$$T(b,c,a) + (-1)^c T(c,a,b) + (-1)^b T(a,b,c) = 2N(b,c,a), \quad (4)$$
$$T(c,a,b) + (-1)^a T(a,b,c) + (-1)^c T(b,c,a) = 2N(c,a,b). \quad (5)$$

If $a+b+c$ is odd, then multiplying (4) by $(-1)^a$ and multiplying (5) by $(-1)^b$ and adding the two equations shows that

$$2T(a,b,c) = (-1)^b 2N(b,c,a) + (-1)^a 2N(c,a,b),$$

as required. $\square$

To avoid confusion with the Euler double zeta function $\zeta(s,t) = \sum_{m,n=1}^{\infty} \frac{1}{m^t(m+n)^s}$, we denote the Hurwitz zeta function by

$$H(z,\alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^z}, \quad \Re z > 1, \quad 0 < \alpha \leq 1.$$

Espinosa and Moll (2006): If $a,b,c \in \mathbb{R} \setminus \mathbb{Z}^+$, then

$$T(a,b,c) = 4\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}{(2\pi)^{3-a-b-c}} \times \Big\{ \big[J(c,a,b) + J(c,b,a)\big] \cos\tfrac{1}{2}\pi(a-b) - \big[I(a,b,c) + J(a,b,c)\big] \cos\tfrac{1}{2}\pi(a+b) \Big\},$$

where

$$I(a,b,c) := \int_0^1 H(1-a,\alpha)H(1-b,\alpha)H(1-c,\alpha)\,d\alpha$$
$$J(a,b,c) := \int_0^1 H(1-a,\alpha)H(1-b,\alpha)H(1-c,1-\alpha)\,d\alpha.$$

Mordell-Tornheim Sums

Let r and w be positive integers, and let $s_1, \ldots, s_r$ and s be complex numbers satisfying $s_1 + \cdots + s_r + s = w$.

A Mordell-Tornheim sum of depth r and weight w is a multiple series of the form

$$T(s_1, \ldots, s_r; s) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s}. \qquad (6)$$

Denote the real part of s by σ, and the real part of s_j by σ_j for $1 \le j \le r$. Since (6) remains unchanged if the arguments $s_1, \ldots, s_r$ are permuted, we may as well suppose that they are arranged in order of increasing real part. Then $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_r$, and it can be shown that the series (6) is absolutely convergent if

$$\sigma + \sum_{j=1}^{k} \sigma_j > k$$

for each $k = 1, 2, \ldots, r$.

We call (6) a Mordell-Tornheim zeta value in the case when the arguments are all integers.

These were first investigated by Tornheim (1950) in the case $r = 2$, and later by Mordell (1958) and Hoffman (1992) with each $s_j = 1$.

Recently, Tsumura (2005) proved that when $r \ge 2$ and $r + w$ is odd, the Mordell-Tornheim zeta value (6) can be expressed in terms of Mordell-Tornheim zeta values of depth less than r:

Theorem 4 (Tsumura) *Every Mordell-Tornheim zeta value of depth at least two and with weight and depth of opposite parity can be expressed as a rational linear combination of products of Mordell-Tornheim zeta values of lower depth.*

Here, we prove a similar result, but instead the Mordell-Tornheim zeta value is expressed in terms of what are now commonly referred to as *multiple zeta values* of lower depth.

A multiple zeta sum of depth r and weight w is a multiple series of the form

$$\zeta(s_1,\ldots,s_r) := \sum_{n_1>\cdots>n_r>0} \prod_{j=1}^{r} n_j^{-s_j}, \tag{7}$$

with $s_1+\cdots+s_r = w$. The sum is over all positive integers $n_1,\ldots,n_r$ such that $n_j > n_{j+1}$ for $1 \le j \le r-1$.

It can be shown that the series (7) is absolutely convergent if the partial sums of the real parts of the arguments satisfy

$$\sum_{j=1}^{k} \Re(s_j) > k$$

for each $k = 1,2,\ldots,r$. Here, we do *not* assume the real parts are arranged in increasing order.

By expressing an *arbitrary* Mordell-Tornheim zeta value (with no parity restriction on the weight and depth) in terms of multiple zeta values of the same weight and depth, we can prove the following result.

Theorem 5 *Every Mordell-Tornheim zeta value of depth at least two and with weight and depth of opposite parity can be expressed as a rational linear combination of products of multiple zeta values of lower depth.*

The case $r = 2$ of Theorem 5 was proved by Tornheim (1950).

Note that by Tsumura's Theorem 4, it suffices to prove the following result.

Theorem 6 *Every Mordell-Tornheim zeta value of depth r and weight w can be expressed as a rational linear combination of multiple zeta values of depth r and weight w.*

Theorem 6 shows that the study of Mordell-Tornheim zeta values reduces to the study of multiple zeta values.

Key to our proof of Theorem 6 is the following partial fraction decomposition.

Lemma 7 *Let r and $s_1, s_2, \ldots, s_r \in \mathbb{Z}^+$, and let $x_1, x_2, \ldots, x_r$ be non-zero real numbers such that $x := x_1 + x_2 + \cdots + x_r \neq 0$. Then*

$$\prod_{j=1}^{r} x_j^{-s_j} = \sum_{j=1}^{r} \left(\prod_{\substack{k=1 \\ k\neq j}}^{r} \sum_{a_k=0}^{s_k-1} \right) M_j x^{-s_j-A_j} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{a_k-s_k},$$

where the multinomial coefficient

$$M_j := \frac{(s_j + A_j - 1)!}{(s_j-1)!} \prod_{\substack{k=1 \\ k\neq j}}^{r} \frac{1}{a_k!} \qquad \text{and} \qquad A_j := \sum_{\substack{k=1 \\ k\neq j}}^{r} a_k.$$

Proof. Applying the partial differential operator

$$\prod_{n=1}^{r} \frac{1}{(s_n-1)!} \left(-\frac{\partial}{\partial x_n} \right)^{s_n-1}$$

to both sides of the trivial identity

$$\prod_{j=1}^{r} x_j^{-1} = \sum_{j=1}^{r} x^{-1} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{-1}, \qquad x := \sum_{j=1}^{r} x_j$$

yields

$$\begin{aligned}\prod_{j=1}^{r} x_j^{-s_j} &= \sum_{j=1}^{r} \left\{ \prod_{\substack{n=1 \\ n\neq j}}^{r} \frac{1}{(s_n-1)!} \left(-\frac{\partial}{\partial x_n} \right)^{s_n-1} \right\} x^{-s_j} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{-1} \\ &= \sum_{j=1}^{r} \left(\prod_{\substack{k=1 \\ k\neq j}}^{r} \sum_{a_k=0}^{s_k-1} \right) \left(\frac{(s_j+A_j-1)!}{(s_j-1)!} \prod_{\substack{k=1 \\ k\neq j}}^{r} \frac{1}{a_k!} \right) x^{-s_j-A_j} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{a_k-s_k},\end{aligned}$$

as claimed. ∎

Proof of Theorem 6. For $1 \le l \le r-1$, let

$$T_l(s_1,\ldots,s_r) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \left(\prod_{k=1}^{l} m_k^{-s_k}\right)\left(\prod_{k=l+1}^{r} n_k^{-s_k}\right), \quad n_k := \sum_{j=1}^{k} m_j.$$

In Lemma 7, let $x_j = m_j$, multiply both sides by n_r^{-s} and sum over all positive integers m_j for $1 \le j \le r$. We find that

$$T(s_1,\ldots,s_r;s) = \sum_{j=1}^{r} \left(\prod_{\substack{k=1\\k\neq j}}^{r} \sum_{a_k=0}^{s_k-1}\right) M_j \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} n_r^{-s-s_j-A_j} \prod_{\substack{k=1\\k\neq j}}^{r} m_k^{a_k-s_k}$$

$$= \sum_{j=1}^{r} \left(\prod_{\substack{k=1\\k\neq j}}^{r} \sum_{a_k=0}^{s_k-1}\right) M_j T_{r-1}(\operatorname*{Cat}_{\substack{k=1\\k\neq j}}^{r}\{s_k-a_k\}, s+s_j+A_j), \tag{8}$$

where $\operatorname*{Cat}_{\substack{k=1\\k\neq j}}^{r}\{t_k\}$ abbreviates the concatenated argument sequence

$$t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{r-1}, t_r.$$

Note that (8) preserves the weight.

Now apply Lemma 7 with $r = l$, $x_j = m_j$, multiply both sides by

$$\prod_{k=l+1}^{r} n_k^{-s_k}$$

and sum over all positive integers m_j for $1 \le j \le r$. We find that

$$T_l(s_1,\ldots,s_r) = \sum_{j=1}^{l} \left(\prod_{\substack{k=1\\k\neq j}}^{l} \sum_{a_k=0}^{s_k-1}\right) M_j \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \left(\prod_{\substack{k=1\\k\neq j}}^{l} m_k^{a_k-s_k}\right)$$

$$\times\, n_l^{-s_j-A_j} \prod_{k=l+1}^{r} n_k^{-s_k}$$

$$= \sum_{j=1}^{l} \left(\prod_{\substack{k=1\\k\neq j}}^{l} \sum_{a_k=0}^{s_k-1}\right) M_j T_{l-1}(\operatorname*{Cat}_{\substack{k=1\\k\neq j}}^{l}\{s_k-a_k\}, s_j+A_j, \operatorname*{Cat}_{k=l+1}^{r} s_k). \tag{9}$$

Since $\sum_{\substack{k=1\\k\neq j}}^{l}(s_k-a_k)+s_j+A_j+\sum_{k=l+1}^{r} s_k = \sum_{k=1}^{r} s_k$, the weight is preserved in (9).

Finally, since

$$T_1(s_1,\ldots,s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1} \prod_{k=2}^{r} n_k^{-s_k}$$

$$= \sum_{n_r>\cdots>n_1>0} \prod_{k=1}^{r} n_k^{-s_k} = \zeta(s_r,\ldots,s_1),$$

by induction the proof is complete. ∎

Corollary 2 *Let $r-1$ and s_j-1 be positive integers for $1 \le j \le r$. Let M_j and A_j be as in Lemma 7 and let T_{r-1} be as in the proof of Theorem 6. Then*

$$\prod_{j=1}^{r} \zeta(s_j) = \sum_{j=1}^{r} \left(\prod_{\substack{k=1\\k\neq j}}^{r} \sum_{a_k=0}^{s_k-1} \right) M_j T_{r-1}(\operatorname*{Cat}_{\substack{k=1\\k\neq j}}^{r}\{s_k-a_k\}, s_j+A_j).$$

Proof. Sum both sides of Lemma 7 over all positive integers $x_1,\ldots,x_r$. ∎

Note that when $r=2$, Corollary 2 reduces to Euler's decomposition

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a).$$

Recall a well-known theorem on multiple zeta values due to Tsumura (2002) and subsequently given an independent proof by Ihara, Kaneko and Zagier (2006):

Theorem 8 (Tsumura) *Every multiple zeta value of depth at least two and with weight and depth of opposite parity can be expressed as a rational linear combination of products of multiple zeta values of lower depth.*

We see that Theorem 5 can also be proved by combining Theorems 6 and 8.

Signed and Unsigned q-Analogs

Henceforth assume q is real and $q > 1$. The q-analog of a positive integer n is

$$[n]_q := \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1}.$$

Let k be a positive integer, let $s_1, s_2, \ldots, s_k$ be real numbers, and let $\sigma_1, \sigma_2, \ldots, \sigma_k \in \{-1, 1\}$. Define the q-Euler sum

$$\zeta_q[s_1, s_2, \ldots, s_k; \sigma_1, \sigma_2, \ldots, \sigma_k] := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} \frac{\sigma_j^{n_j} q^{(s_j - 1)n_j}}{[n_j]_q^{s_j}}.$$

If each $\sigma_j = 1$, we recover the multiple q-zeta function $\zeta_q[s_1, s_2, \ldots, s_k]$.

Let $\sigma, \tau \in \{-1, 1\}$. Define signed and unsigned q-analogs of the signed and unsigned Tornheim double series by

$$T[r, s, t; \sigma, \tau] := \sum_{u,v=1}^{\infty} \frac{\sigma^u \tau^v q^{(r+t-1)u + (s+t-1)v}}{[u]_q^r [v]_q^s [u+v]_q^t},$$

and also

$$\varphi[s; \sigma] := \sum_{n=1}^{\infty} \frac{(n-1)\sigma^n q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{n\sigma^n q^{(s-1)n}}{[n]_q^s} - \zeta_q[s; \sigma].$$

For convenience, we combine signs and exponents into a single list by writing s_j if $\sigma_j = 1$ and $\overline{s_j}$ if $\sigma_j = -1$. For example,

$$\varphi[s; 1] = \varphi[s], \quad \varphi[s; -1] = \varphi[\bar{s}], \quad T[r, s, t; 1, -1] = T[r, \bar{s}, t].$$

We also employ the notation

$$\binom{z}{a,\, b} := \binom{z}{a}\binom{z-a}{b} = \binom{z}{b}\binom{z-b}{a}$$

for the trinomial coefficient, in which a, b are nonnegative integers, and which reduces to $z!/a!b!(z-a-b)!$ if z is also an integer exceeding $a + b$.

Theorem 9 *Let* $r, s \in \mathbb{Z}^+$ *and let* $t \in \mathbb{R}$. *Then*

$$T[r,s,t] = \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b\zeta_q[s+t+a, r-a-b]$$
$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}(1-q)^b\zeta_q[r+t+a, s-a-b]$$
$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j\varphi[r+s+t-j],$$

$$T[\bar{r},\bar{s},t] = \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b\zeta_q[\overline{s+t+a}, r-a-b]$$
$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}(1-q)^b\zeta_q[\overline{r+t+a}, s-a-b]$$
$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j\varphi[\overline{r+s+t-j}].$$

Taking the limit as $q \to 1+$ in Theorem 9 and noting the restrictions on r, s and t now needed for convergence yields the following

Corollary 3 *Let r and s be positive integers and let t be a real number. If $r+t>1$ and $s+t>1$, then*

$$T(r,s,t) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(s+t+a, r-a)$$
$$+\sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+t+a, s-a);$$

if $r+t>0$ and $s+t>0$, then

$$T(\bar{r},\bar{s},t) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+t+a}, r-a)$$
$$+\sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(\overline{r+t+a}, s-a).$$

Putting $t = 0$ in Theorem 9 yields

Corollary 4 *If r and s are positive integers, then*

$$\zeta_q[r]\zeta_q[s] = \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\zeta_q[s+a, r-a-b]$$
$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{r-1}\binom{r-1}{b}(1-q)^b\zeta_q[r+a, s-a-b]$$
$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\, s-j}(1-q)^j\varphi[r+s-j];$$

$$\zeta_q[\overline{r}]\zeta_q[\overline{s}] = \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\zeta_q[\overline{s+a}, r-a-b]$$
$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{r-1}\binom{r-1}{b}(1-q)^b\zeta_q[\overline{r+a}, s-a-b]$$
$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\, s-j}(1-q)^j\varphi[\overline{r+s-j}].$$

Taking the limit as $q \to 1+$ in Corollary 2 and noting the additional restrictions needed on r and s to guarantee convergence in this case yields the following decomposition formulas, the first of which was known to Euler.

Corollary 5 *If $r-1$ and $s-1$ are positive integers, then*

$$\zeta(r)\zeta(s) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(s+a, r-a) + \sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+a, s-a);$$

if r and s are positive integers, then

$$\zeta(\overline{r})\zeta(\overline{s}) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+a}, r-a) + \sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(\overline{r+a}, s-a);$$

if $r-1$ and s are positive integers, then

$$\zeta(r)\zeta(\overline{s}) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+a}, \overline{r-a}) + \sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+a, \overline{s-a}).$$

Proof of Theorem 9

The key ingredient is the following partial fraction decomposition.

Lemma 10 *If r, s, u, and v are all positive integers, then*

$$\frac{1}{[u]_q^r[v]_q^s} = \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\, b}\frac{(1-q)^b q^{(s-1-b)u+av}}{[u]_q^{r-a-b}[u+v]_q^{s+a}}$$

$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\, b}\frac{(1-q)^b q^{au+(r-1-b)v}}{[v]_q^{s-a-b}[u+v]_q^{r+a}}$$

$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\, s-j}\frac{(1-q)^j q^{(s-j)u+(r-j)v}}{[u+v]_q^{r+s-j}}.$$

Proof. Let $x, y \in \mathbb{R}$ be such that $xy \neq 0$ and $x+y+(q-1)xy \neq 0$. Observe that if we apply the partial differential operator

$$\frac{1}{(r-1)!}\left(-\frac{\partial}{\partial x}\right)^{r-1}\frac{1}{(s-1)!}\left(-\frac{\partial}{\partial y}\right)^{s-1}$$

to both sides of the identity

$$\frac{1}{xy} = \frac{1}{x+y+(q-1)xy}\left(\frac{1}{x}+\frac{1}{y}+q-1\right),$$

then we obtain the identity

$$\frac{1}{x^ry^s} = \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\, b}\frac{(1-q)^b(1+(q-1)y)^a(1+(q-1)x)^{s-1-b}}{x^{r-a-b}(x+y+(q-1)xy)^{s+a}}$$

$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\, b}\frac{(1-q)^b(1+(q-1)x)^a(1+(q-1)y)^{r-1-b}}{y^{s-a-b}(x+y+(q-1)xy)^{r+a}}$$

$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\, s-j}\frac{(1-q)^j(1+(q-1)y)^{r-j}(1+(q-1)x)^{s-j}}{(x+y+(q-1)xy)^{r+s-j}}.$$

Now let $x = [u]_q$, $y = [v]_q$ and note that then $1+(q-1)x = q^u$, $1+(q-1)y = q^v$ and $x+y+(q-1)xy = [u+v]_q$. □

To prove Theorem 9, multiply both sides of Lemma 10 by

$$\frac{\sigma^u \tau^v q^{(r+t-1)u+(s+t-1)v}}{[u+v]_q^t}$$

and sum over all ordered pairs of positive integers (u,v) to obtain

$$T[r,s,t;\sigma,\tau]$$
$$= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b \sum_{u,v=1}^{\infty}\frac{\sigma^u\tau^v q^{(r-a-b-1)u} q^{(s+t+a-1)(u+v)}}{[u]_q^{r-a-b}[u+v]_q^{s+t+a}}$$
$$+ \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}(1-q)^b \sum_{u,v=1}^{\infty}\frac{\sigma^u\tau^v q^{(s-a-b-1)v} q^{(r+t+a-1)(u+v)}}{[v]_q^{s-a-b}[u+v]_q^{r+t+a}}$$
$$- \sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j \sum_{u,v=1}^{\infty}\frac{\sigma^u\tau^v q^{(r+s+t-j-1)(u+v)}}{[u+v]_q^{r+s+t-j}}.$$

It follows that

$$T[r,s,t;\sigma,\sigma]$$
$$= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b \sum_{m>n>0}\frac{\sigma^m q^{(s+t+a-1)m} q^{(r-a-b-1)n}}{[m]_q^{s+t+a}[n]_q^{r-a-b}}$$
$$+ \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}(1-q)^b \sum_{m>n>0}\frac{\sigma^m q^{(r+t+a-1)m} q^{(s-a-b-1)n}}{[m]_q^{r+t+a}[n]_q^{s-a-b}}$$
$$- \sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j \sum_{m>n>0}\frac{\sigma^m q^{(r+s+t-j-1)m}}{[m]_q^{r+s+t-j}}$$
$$= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b \zeta_q[s+t+a,r-a-b;\sigma,1]$$
$$+ \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}(1-q)^b \zeta_q[r+t+a,s-a-b;\sigma,1]$$
$$- \sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j \varphi[r+s+t-j;\sigma].$$

∎