A q-analog of Euler's reduction formula for the double zeta function

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Double Zeta Function

Definition 1. The Riemann zeta function is defined in the right half-plane $\{s \in \mathbf{C} : \Re(s) > 1\}$ by the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Definition 2. The double zeta function is a function of two complex variables defined by the double series

$$\zeta(s,t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \qquad \Re(s) > 1, \qquad \Re(s+t) > 2.$$

The double zeta function was first studied by Euler in response to a letter from Goldbach in 1742.

Theorem 1 (Euler) *If $s-1$ and $t-1$ are positive integers, then the* <u>*decomposition formula*</u>

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a)$$

holds.

A combinatorial proof of Euler's decomposition formula based on the simplex integral representations

$$\zeta(s) = \int_{1>x_1>\cdots>x_s>0} \left(\prod_{i=1}^{s-1} \frac{dx_i}{x_i} \right) \frac{dx_s}{1-x_s},$$

$$\zeta(s,t) = \int_{1>x_1>\cdots>x_{s+t}>0} \left(\prod_{i=1}^{s-1} \frac{dx_i}{x_i} \right) \frac{dx_s}{1-x_s} \left(\prod_{i=s+1}^{s+t-1} \frac{dx_i}{x_i} \right) \frac{dx_{s+t}}{1-x_{s+t}},$$

and the shuffle multiplication rule satisfied by such integrals is given in a 1998 EJC paper of J. Borwein, D. Bradley, D. Broadhurst and P. Lisoněk.

Euler's Decomposition Formula a.k.a. Shuffle Product

Previously, in their paper

Explicit Evaluation of Euler Sums, *Proc. Edinburgh Math. Soc.* **38** (1995), 277-294

D. Borwein, J. Borwein and R. Girgensohn gave an algebraic (integral-free) proof of Euler's decomposition formula (for integers $s > 1$, $t > 1$)

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a)$$

by summing Nielsen's partial fraction decomposition

$$\frac{1}{x^s (c-x)^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a} c^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{c^{s+a} (c-x)^{t-a}}$$

over appropriately chosen integers x and c.

Euler's Reduction Formula

In his 1742 letter to Euler, Goldbach actually proposed the problem of evaluating $\zeta(s,t)$ for integers $s > 1$ and $t > 0$.

Calculating several examples led Euler to infer a closed form evaluation in terms of values of the Riemann zeta function, in the case when the two arguments have opposite parity:

Let $s-1$ and $t-1$ be positive integers with $s+t$ odd and let $2h = \max(s,t)$. Then the reduction formula

$$\zeta(s,t) = \frac{1}{2}\Big((1+(-1)^s\Big)\zeta(s)\zeta(t) + \frac{1}{2}\left[(-1)^s\binom{s+t}{s} - 1\right]\zeta(s+t)$$
$$+ (-1)^{s+1}\sum_{k=1}^{h}\left[\binom{s+t-2k-1}{t-1} + \binom{s+t-2k-1}{s-1}\right]\zeta(2k)\zeta(s+t-2k)$$

holds.

Euler apparently did not give a proof of his reduction formula.

A proof was later given by D. Borwein, J. Borwein and R. Girgensohn in their 1995 *Proc. Edinburgh Math. Soc.* paper, Explicit Evaluation of Euler Sums.

Using the reflection formula (a.k.a. stuffle product)

$$\zeta(s)\zeta(t) = \zeta(s,t) + \zeta(s+t) + \zeta(t,s)$$

obtained by grouping the indices (n,k) in the series on the left according to whether $n > k$, $n = k$ or $n < k$, and Euler's decomposition formula

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a),$$

they derived a system of linear equations for the values of $\zeta(s,t)$ with $s+t$ fixed but odd, and then solved the resulting system.

When $s+t$ is even, the system does not in general have a unique solution.

Degenerate Case: Euler's Convolution Formula

Recall that Euler's reduction formula

$$\zeta(s,t) = \frac{1}{2}\Big((1+(-1)^s\Big)\zeta(s)\zeta(t) + \frac{1}{2}\left[(-1)^s\binom{s+t}{s} - 1\right]\zeta(s+t)$$
$$+(-1)^{s+1}\sum_{k=1}^{h}\left[\binom{s+t-2k-1}{t-1} + \binom{s+t-2k-1}{s-1}\right]\zeta(2k)\zeta(s+t-2k),$$

is valid for integers $s > 1$ and $t > 1$ with $s + t$ odd and $2h = \max(s,t)$.

If we interpret $\zeta(1) = 0$ the formula gives true results also when $t = 1$ and s is even, but this case is subsumed by another formula of Euler, namely the convolution formula

$$\zeta(s,1) = \frac{1}{2}s\,\zeta(s+1) - \frac{1}{2}\sum_{k=2}^{s-1}\zeta(k)\zeta(s+1-k),$$

which is valid for all integers $s > 1$, not just even $s > 1$.

Multiple Zeta Values

Definition. (Michael Hoffman 1992, Don Zagier 1994)

$$\zeta(s_1, \ldots, s_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}.$$

The multiple series is absolutely convergent if

$$\sum_{j=1}^{n} \Re(s_j) > n, \qquad n = 1, 2, \ldots, m.$$

Example. With $m = 2$, $s_1 = s$ and $s_2 = 1$, we have Euler's convolution formula

$$2\zeta(s, 1) = s\zeta(s + 1) - \sum_{j=1}^{s-2} \zeta(s - j)\zeta(j + 1),$$

where $2 \leq s \in \mathbf{Z}$. In particular, $\zeta(2, 1) = \zeta(3)$.

Period One

For all non-negative integers n,

$$\zeta(\{2\}^n) := \zeta(\underbrace{2,2,\ldots,2}_{n}) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\{4\}^n) := \zeta(\underbrace{4,4,\ldots,4}_{n}) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!},$$

$$\zeta(\{6\}^n) := \zeta(\underbrace{6,6,\ldots,6}_{n}) = \frac{6(2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(\{8\}^n) := \zeta(\underbrace{8,8,\ldots,8}_{n}) = \frac{8(2\pi)^{8n}}{(8n+4)!}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1-\frac{1}{\sqrt{2}}\right)^{4n+2}\right\}.$$

More generally, let $k \in \mathbf{Z}^+$ and $\omega := e^{i\pi/k}$. Then

$$\sum_{n=0}^{\infty}(-1)^n x^{2kn}\zeta(\{2k\}^n) = \prod_{j=0}^{k-1}\frac{\sin(\pi x\omega^j)}{\pi x\omega^j}.$$

Period Two

For all non-negative integers n,

$$\zeta(\{3,1\}^n) = 4^{-n}\zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!},$$

$$\begin{aligned}\zeta(3,\{1,3\}^n) &= 4^{-n}\sum_{k=0}^{n}\zeta(4k+3)\zeta(\{4\}^{n-k}) \\ &= \sum_{k=0}^{n}\frac{2\pi^{4k}}{(4k+2)!}\left(-\frac{1}{4}\right)^{n-k}\zeta(4n-4k+3),\end{aligned}$$

$$\begin{aligned}\zeta(2,\{1,3\}^n) = 4^{-n}\sum_{k=0}^{n}(-1)^k\zeta(\{4\}^{n-k})\Bigg\{&(4k+1) \\ &\times\zeta(4k+2) - 4\sum_{j=1}^{k}\zeta(4j-1)\zeta(4k-4j+3)\Bigg\}.\end{aligned}$$

q-Calculus

Let $f : \mathbf{R} \to \mathbf{R}$ and let $q \neq 1$.

Definition. The q-*difference* operator d_q is defined by

$$(d_q f)(x) = f(qx) - f(x).$$

Example. For the identity map ι, since $\iota(x) = x$, we have

$$(d_q \iota)(x) = qx - x = (q-1)x.$$

Definition. The q-*derivative* operator D_q is defined by

$$(D_q f)(x) = \frac{(d_q f)(x)}{(d_q \iota)(x)} = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0.$$

Notation. If $y = f(x)$, then

$$\frac{d_q y}{d_q x} = (D_q f)(x).$$

The q-Derivative

Recall that if $y = f(x)$, then the q-derivative is

$$\frac{d_q y}{d_q x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

If f is differentiable at $x \neq 0$, then we recover the ordinary derivative $f'(x)$ from the q-derivative in the limit as $q \to 1$.

To see this, substitute $q = 1 + h/x$. Then

$$\lim_{q \to 1} \frac{f(qx) - f(x)}{(q-1)x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The q-Product Rule

Recall that if $y = f(x)$, then the q-derivative is

$$\frac{d_q y}{d_q x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Therefore, the q-derivative of the product of two functions f, g is given by

$$(D_q fg)(x) = \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x}$$

$$= \frac{f(qx)[g(qx) - g(x)] + [f(qx) - f(x)]g(x)}{(q-1)x}$$

$$= f(qx)(D_q g)(x) + g(x)(D_q f)(x).$$

The q-Integral

J. Thomae (1869): Let $0 < q < 1$. Then

$$\int_0^1 f(t)\, d_q t := (1-q) \sum_{n=0}^{\infty} q^n f(q^n).$$

F. H. Jackson (1910):

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,$$

$$\int_0^b f(t)\, d_q t := (1-q) \sum_{n=0}^{\infty} b q^n f(b q^n).$$

If $f : [0..b] \to \mathbf{R}$ is continuous then

$$\lim_{q \to 1} \int_0^b f(t)\, d_q t = \int_0^b f(t)\, dt.$$

Fundamental Theorem of q-Calculus

Suppose $f : (0, b] \to \mathbf{R}$ and $0 < x \leq b$.

The Jackson q-integral of f is defined by

$$\int_0^x f(t)\, d_q t := (1-q) \sum_{j=0}^{\infty} x q^j f(x q^j).$$

If there exists $0 \leq \alpha < 1$ such that $|f(t) t^{\alpha}|$ is bounded on $(0, b]$, then the integral converges to a function $F(x)$ on $(0, b]$.

Additionally, F is a q-antiderivative of f:

$$D_q F(x) := \frac{F(qx) - F(x)}{(q-1)x} = f(x), \qquad 0 < x \leq b.$$

Note that

$$\lim_{q \to 1} D_q F(x) = F'(x), \qquad \text{and} \qquad \lim_{q \to 1} \int_0^x f(t)\, d_q t = \int_0^x f(t)\, dt.$$

The q-analog of α

Recall that if $y = f(x)$, then

$$\frac{d_q\, y}{d_q\, x} = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Example. Let $\alpha \in \mathbf{R}$ and $y = x^\alpha$. Then

$$\frac{d_q\, y}{d_q\, x} = \frac{(qx)^\alpha - x^\alpha}{(q-1)x} = \left(\frac{q^\alpha - 1}{q-1}\right) x^{\alpha-1},$$

$$\lim_{q\to 1} \frac{d_q\, y}{d_q x} = \frac{dy}{dx} = \alpha\, x^{\alpha-1}.$$

Definition. Let $\alpha \in \mathbf{R}$ and $q \neq 1$. The q-*analog* of α is

$$[\alpha]_q := \frac{q^\alpha - 1}{q-1}.$$

Multiple q-Zeta Values

M. Kaneko (2003) investigated analytic properties of the Riemann q-zeta function

$$\zeta[s] := \sum_{k=1}^{\infty} \frac{q^{tk}}{[k]_q^s}, \qquad t = s - 1.$$

This suggested the definition

$$\zeta[s_1, \ldots, s_m] := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Alternatively, we could let

$$Z[s_1, \ldots, s_m] := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

If $\vec{s} = (s_1, \ldots, s_m)$, then

$$\zeta[\vec{s}; q] = q^{|\vec{s}|} Z[\vec{s}; 1/q], \qquad |\vec{s}| := \sum_{j=1}^{m} s_j.$$

Period-1 Sums Reduce

Theorem 2 *If n is a positive integer and $s > 1$, then*

$$n\zeta[\{s\}^n] = \sum_{k=1}^{n} (-1)^{k+1} \zeta[\{s\}^{n-k}] \sum_{j=0}^{k-1} \binom{k-1}{j} (1-q)^j \zeta[ks-j].$$

Example 1 *With $n = 2$, we get*

$$2\zeta[s,s] = \zeta[s]\zeta[s] - \Big(\zeta[2s] + (1-q)\zeta[2s-1]\Big).$$

Corollary 1 *If n is a positive integer and $s > 1$, then*

$$n\zeta(\{s\}^n) = \sum_{k=1}^{n} (-1)^{k+1} \zeta(\{s\}^{n-k}) \zeta(ks).$$

Sums Over Permutations

Let $\mathfrak{S}_n$ denote the group of $n!$ permutations of $\langle n \rangle = \{1, 2, \ldots, n\}$.

Theorem 3 *Let n be a positive integer, and let $s_j > 1$ for $1 \le j \le n$. Then*

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta\Big[\operatorname*{Cat}_{j=1}^{n} s_{\sigma(j)}\Big] = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{k=1}^{|\mathcal{P}|} (|P_k| - 1)! \times \sum_{\nu_k=0}^{|P_k|-1} \binom{|P_k| - 1}{\nu_k} (1-q)^{\nu_k} \zeta[p_k - \nu_k],$$

where the outer sum on the right is over all unordered set partitions $\mathcal{P} = \{P_1, \ldots, P_m\}$ of $\langle n \rangle$, $1 \le m = |\mathcal{P}| \le n$, and $p_k = \sum_{j \in P_k} s_j$.

Corollary 2 (M. Hoffman)

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta\Big(\operatorname*{Cat}_{j=1}^{n} s_{\sigma(j)}\Big) = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{P \in \mathcal{P}} (|P| - 1)!\, \zeta\Big(\sum_{j \in P} s_j\Big).$$

Parity Reduction

Theorem 4 *Let $m \in \mathbf{Z}^+$ and let $s_1, \ldots, s_m$ be real numbers with $s_1 > 1$, $s_m > 1$, and $s_j \geq 1$ for $1 < j < m$. Then*

$$\zeta\Big[\operatorname*{\mathbf{Cat}}_{k=1}^{m} s_k\Big] + (-1)^m \zeta\Big[\operatorname*{\mathbf{Cat}}_{k=1}^{m} s_{m-k+1}\Big]$$

can be expressed as a $\mathbf{Z}[q]$-linear combination of multiple q-zeta values of depth less than m.

In other words, the coefficients in the linear combination are polynomials in q with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

A Double Generating Function

Theorem 5

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^{m+1}v^{n+1}\zeta[m+2,\{1\}^n]$$
$$= 1 - \exp\left\{\sum_{k=2}^{\infty}\left\{u^k + v^k - \Big(u+v+(1-q)uv\Big)^k\right\} \times \frac{1}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\}.$$

Corollary 3 *If* $0 \le m, n \in \mathbf{Z}$, *then*

$$\zeta[m+2,\{1\}^n] = \zeta[n+2,\{1\}^m].$$

Corollary 4 (q-Euler convolution) *Let* $0 \le m \in \mathbf{Z}$. *Then*

$$2\zeta[m+2,1] = (m+2)\zeta[m+3] + (1-q)\,m\,\zeta[m+2] - \sum_{k=2}^{m+1}\zeta[m+3-k]\,\zeta[k].$$

The proof of Theorem 5 makes essential use of the basic hypergeometric function

$$ {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}\,\middle|\, x\right] = 1 + \sum_{n=1}^{\infty} x^n \prod_{k=0}^{n-1} \frac{(1-q^{a+k})(1-q^{b+k})}{(1-q^{c+k})(1-q^{1+k})}, \qquad |x| < 1. $$

Routine series manipulations reveal that

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta[m+2,\{1\}^n] $$
$$ = 1 - {}_2\phi_1\left[\begin{matrix} q^{-y}, q^x \\ q^{1+x} \end{matrix}\,\middle|\, q^{1+y}\right]. $$

Heine's q-analog

$$ {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix} \,\middle|\, q^{c-a-b}\right] = \frac{\Gamma_q(c)\Gamma_q(c-a-b)}{\Gamma_q(c-a)\Gamma_q(c-b)} $$

of Gauss's ${}_2F_1$ summation formula then gives

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta[m+2,\{1\}^n] = 1 - \frac{\Gamma_q(1+x)\Gamma_q(1+y)}{\Gamma_q(1+x+y)}, $$

where for $0 < q < 1$,

$$ \Gamma_q(1+x) := (1-q)^{-x}\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{n+x})}, $$

is the q-analog of Euler's gamma function.

If $\vec{s} = (s_1, \ldots, s_m)$ define

$$\begin{aligned} \text{weight}(\vec{s}) &:= |\vec{s}| = \sum_{j=1}^{m} s_j, \\ \text{depth}(\vec{s}) &:= m, \\ \text{height}(\vec{s}) &:= \#\{j : s_j \geq 2\}. \end{aligned}$$

Theorem 6 (J. Okuda & Y. Takeyama)

$$1 + (w - uv) \sum_{s,m,h \geq 0} u^{s-m-h} v^{m-h} w^{h-1} \sum_{\substack{\text{weight}(\vec{s})=s \\ \text{depth}(\vec{s})=m \\ \text{height}(\vec{s})=h}} \zeta[\vec{s}]$$

$$= \exp\left\{ \sum_{k=2}^{\infty} \left(u^k + v^k - \alpha^k - \beta^k\right) \frac{1}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta[j] \right\},$$

where α and β satisfy the equations

$$\alpha + \beta = u + v + (q-1)(w - uv), \qquad \alpha\beta = w.$$

Theorem 5 is case $w = 0$ of Theorem 6.

The Simplex Integral

M. Kontsevich: If $s_1, \ldots, s_m \in \mathbf{Z}^+$, then

$$\zeta(s_1, \ldots, s_m) = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0,$$

and is abbreviated (D. Broadhurst) by

$$\int_0^1 \prod_{k=1}^{m} A^{s_k - 1} B, \qquad A = \frac{dt}{t}, \quad B = \frac{dt}{1-t}.$$

Example

$$
\begin{aligned}
\zeta(2,1) &= \sum_{n>m>0} n^{-2} m^{-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k+j)^{-2} k^{-1} \\
&= \sum_{k=1}^{\infty} k^{-1} \sum_{j=1}^{\infty} (k+j)^{-1} \int_0^1 t^{k+j-1}\, dt \\
&= \sum_{k=1}^{\infty} k^{-1} \int_0^1 t^{-1} \sum_{j=1}^{\infty} \int_0^t u^{k+j-1}\, du\, dt \\
&= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} k^{-1} u^k\, du\, dt \\
&= \int_0^1 t^{-1} \int_0^t (1-u)^{-1} \sum_{k=1}^{\infty} \int_0^u v^{k-1}\, dv\, du\, dt \\
&= \int_{1>t>u>v>0} \frac{dt}{t} \cdot \frac{du}{1-u} \cdot \frac{dv}{1-v} \\
&= \int_0^1 AB^2.
\end{aligned}
$$

The Jackson q-Integral

Suppose $f : (0, b] \to \mathbf{R}$ and $0 < x \leq b$.

Recall the Jackson q-integral of f on the subinterval $(0, x]$ is

$$\int_0^x f(t)\, d_q t := (1 - q) \sum_{j=0}^{\infty} x q^j f(x q^j),$$

and if there exists $0 \leq \alpha < 1$ such that $|f(t)t^{\alpha}|$ is bounded on $(0, b]$, then the integral converges to a function $F(x)$ on $(0, b]$.

Additionally (fundamental theorem of q-calculus), F is a q-antiderivative of f:

$$D_q F(x) := \frac{F(qx) - F(x)}{(q-1)x} = f(x), \quad 0 < x \leq b.$$

The Jackson Simplex Integral

Let $s_1, \ldots, s_m$ are positive integers. Recall:

$$\zeta(s_1, \ldots, s_m) = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{dt_r^{(k)}}{t_r^{(k)}} \right) \frac{dt_{s_k}^{(k)}}{1 - t_{s_k}^{(k)}},$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0.$$

Theorem 7

$$\zeta[s_1, \ldots, s_m] = \int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k - 1} \frac{d_q t_r^{(k)}}{t_r^{(k)}} \right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}},$$

where

$$y_k := \prod_{j=1}^{k} q^{1 - s_j},$$

and the integral is over the same simplex as above.

Duality

Let $a_i, b_i \in \mathbf{Z}^+$ and $k = \sum_{i=1}^{n} (a_i + b_i)$. Then

$$\zeta(a_1+1, \{1\}^{b_1-1}, \ldots, a_n+1, \{1\}^{b_n-1}) = \int_0^1 \prod_{i=1}^{n} A^{a_i} B^{b_i}$$

$$= \int\limits_{1>t_1>\cdots>t_k>0} \prod_{j=1}^{k} f_j(t_j)\, dt_j$$

$$= \int\limits_{1>u_k>\cdots>u_1>0} \prod_{j=1}^{k} f_j(u_j)\, du_j, \qquad u_j = 1 - t_j$$

$$= \int_0^1 \prod_{i=n}^{1} A^{b_i} B^{a_i} = \zeta(b_n+1, \{1\}^{a_n-1}, \ldots, b_1+1, \{1\}^{a_1-1}).$$

Generalized Duality

Definition 1 *Let n and $s_1, \ldots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S(s_1, \ldots, s_n; m) := \sum_{\substack{c_1, \ldots, c_n \geq 0 \\ c_1 + \cdots + c_n = m}} \zeta(s_1 + c_1, \ldots, s_n + c_n).$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \left\{ a_i + 1, \{1\}^{b_i - 1} \right\},$$

$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \left\{ b_i + 1, \{1\}^{a_i - 1} \right\}.$$

Theorem 8 (Y. Ohno) *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have the equality*

$$S(\vec{s}\,; m) = S(\vec{s}\,'; m).$$

Generalized q-Duality

Definition 2 *Let n and $s_1, \ldots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define*

$$S[s_1, \ldots, s_n; m] := \sum_{\substack{c_1,\ldots,c_n \ge 0 \\ c_1+\cdots+c_n=m}} \zeta[s_1 + c_1, \ldots, s_n + c_n].$$

For positive integers a_i and b_i, define the dual argument lists

$$\vec{s} = \operatorname*{Cat}_{i=1}^{n} \left\{a_i + 1, \{1\}^{b_i - 1}\right\}$$
$$\vec{s}\,' = \operatorname*{Cat}_{i=n}^{1} \left\{b_i + 1, \{1\}^{a_i - 1}\right\}.$$

Theorem 9 *For any pair of dual argument lists $\vec{s}$, $\vec{s}\,'$ and any non-negative integer m, we have*

$$S[\vec{s}; m] = S[\vec{s}\,'; m].$$

q-Duality

Corollary 5 *If $\vec{s}$, $\vec{s}\,'$ are dual argument lists, then*

$$\zeta[\vec{s}] = \zeta[\vec{s}\,'].$$

In other words, if $a_i, b_i \in \mathbf{Z}^+$ $(1 \le i \le n)$, then

$$\zeta\Big[\mathop{\mathbf{Cat}}_{i=1}^{n}\Big\{a_i + 1, \{1\}^{b_i - 1}\Big\}\Big] = \zeta\Big[\mathop{\mathbf{Cat}}_{i=n}^{1}\Big\{b_i + 1, \{1\}^{a_i - 1}\Big\}\Big].$$

Proof. Put $m = 0$ in Theorem 9 (generalized q-duality).
□

q-Sum Formula

Definition 3 *Let $t_1, \ldots, t_n$ be positive integers.*

$$\zeta^*[t_1, \ldots, t_n] := \zeta\Big[t_1 + 1, \operatorname*{\mathbf{Cat}}_{j=2}^{n} t_j\Big].$$

Corollary 6 (q-Sum Formula) *For any integers $0 < k \leq n$, we have*

$$\sum_{t_1+t_2+\cdots+t_n=k} \zeta^*[t_1, t_2, \ldots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers $t_1, \ldots, t_n$ with sum equal to k.

Proof. If we take the dual argument lists in the form $\vec{s} = (n+1)$ and $\vec{s}\,' = (2, \{1\}^{n-1})$ and put $m = k - n$, then Theorem 9 (generalized q-duality) states that

$$\zeta[k+1] = \sum_{\substack{c_1,\dots,c_n \ge 0 \\ c_1+\cdots+c_n = k-n}} \zeta\Big[2 + c_2, \operatorname*{\mathbf{Cat}}_{j=2}^{n} \{1 + c_j\}\Big]$$

$$= \sum_{\substack{t_1,\dots,t_n \ge 1 \\ t_1+\cdots+t_n = k}} \zeta\Big[t_1 + 1, \operatorname*{\mathbf{Cat}}_{j=2}^{n} t_j\Big].$$

□

q-Cyclic Sum Formula

Definition 4 *Let $s_j \in \mathbf{Z}^+$ for $1 \le j \le n$ and put $\vec{s} = (s_1, \ldots, s_n)$. Let σ denote the n-cycle $(1\,2 \cdots n)$, and let*

$$\mathcal{C}(\vec{s}) := \{(s_{\sigma^j(1)}, \ldots, s_{\sigma^j(n)}) : 1 \le j \le n\}$$

denote the set of cyclic permutations of $\vec{s}$.

Recall the definition

$$\zeta^*[s_1, \ldots, s_n] := \zeta[s_1 + 1, s_2, \ldots, s_n].$$

Theorem 10 *Let $\vec{s}$ and $\vec{s}\,'$ be dual argument lists. Then*

$$\sum_{\vec{t} \in \mathcal{C}(\vec{s})} \zeta^*[\vec{t}\,] = \sum_{\vec{t} \in \mathcal{C}(\vec{s}\,')} \zeta^*[\vec{t}\,].$$

Reformulation of q-Duality

Let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y.

Let $\mathfrak{h}^0$ denote the subalgebra $\mathbf{Q}1 \oplus x\mathfrak{h}y$. The $\mathbf{Q}$-linear map $\widehat{\zeta}$ is defined on $\mathfrak{h}^0$ by

$$\widehat{\zeta}[1] := \zeta[\,] = 1$$

and

$$\widehat{\zeta}\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}\right] = \zeta\left[\operatorname*{\mathbf{Cat}}_{i=1}^{s}\left\{a_i + 1, \{1\}^{b_i - 1}\right\}\right],$$

for positive integers a_i, b_i $(1 \le i \le s)$.

Let τ be the anti-automorphism of $\mathfrak{h}$ that switches x and y.

Then q-duality simply says that

$$\widehat{\zeta}[\tau w] = \widehat{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.$$

Derivations

Definition 5 (K. Ihara & M. Kaneko) *Define a derivation on* $\mathfrak{h}$ *for each positive integer* n *by*

$$\partial_n(x) = x(x+y)^{n-1}, \qquad \partial_n(y) = -x(x+y)^{n-1}y.$$

Theorem 11 (Ihara & Kaneko) *For all positive integers* n *and words* $w \in \mathfrak{h}^0$*,* $\widehat{\zeta}(\partial_n(w)) = 0$*.*

Theorem 12 (q-Analog) *For all positive integers* n *and words* $w \in \mathfrak{h}^0$*,* $\widehat{\zeta}[\partial_n(w)] = 0$*.*

Theorem 12 is actually *equivalent* to generalized q-duality (Theorem 9).

Proof of Theorem 12

Proof. Let $\sigma = \exp(\Delta)$, $\widetilde{\sigma} = \tau\sigma\tau$,

$$\Delta = \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n, \qquad \partial = \sum_{n=1}^{\infty} \frac{\partial_n}{n} \theta^n.$$

Generalized q-duality (Theorem 9): $\forall w \in \mathfrak{h}^0$,

$$\widehat{\zeta}[\sigma w] = \widehat{\zeta}[\sigma\tau w] = \widehat{\zeta}[\tau\sigma\tau w] \iff (\sigma - \widetilde{\sigma})w \in \ker \widehat{\zeta}.$$

We show that in fact, $(\sigma - \widetilde{\sigma})\mathfrak{h}^0 = \partial\mathfrak{h}^0$.

To prove this, we require the following identity of Ihara and Kaneko.

Proposition 13 $\exp(\partial) = \widetilde{\sigma}\sigma^{-1}$.

To complete the proof of Theorem 12, observe that since

$$\partial = \log\left(\widetilde{\sigma}\sigma^{-1}\right) = \log\left(1 - (\sigma - \widetilde{\sigma})\sigma^{-1}\right)$$

$$= -(\sigma - \widetilde{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n}\left((\sigma - \widetilde{\sigma})\sigma^{-1}\right)^{n-1}\sigma^{-1},$$

and

$$\sigma - \widetilde{\sigma} = \left(1 - \widetilde{\sigma}\sigma^{-1}\right)\sigma = \left(1 - \exp(\partial)\right)\sigma$$
$$= -\partial \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{n!}\sigma,$$

we see that

$$\partial\mathfrak{h}^0 \subseteq (\sigma - \widetilde{\sigma})\mathfrak{h}^0 \quad \text{and} \quad (\sigma - \widetilde{\sigma})\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0.$$

Thus for the kernel of $\widehat{\zeta}$, we have the equivalences

$$\begin{aligned} (\sigma - \widetilde{\sigma})w \in \ker\widehat{\zeta} \quad &\Longleftrightarrow \quad \partial w \in \ker\widehat{\zeta} \\ &\Longleftrightarrow \quad \forall n \in \mathbf{Z}^+,\ \widehat{\zeta}[\partial_n w] = 0. \end{aligned}$$

$\square$

q-analog of Euler's Decomposition Formula

The respective q-analogs of the Riemann and double zeta functions are

$$\zeta[s] = \sum_{n>0} \frac{q^{(s-1)n}}{[n]_q^s} \quad \text{and} \quad \zeta[s,t] = \sum_{n>k>0} \frac{q^{(s-1)n} q^{(k-1)t}}{[n]_q^s [k]_q^t}.$$

We also need the sum

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s].$$

Theorem 14 *If* $s-1$ *and* $t-1$ *are positive integers, then*

$$\zeta[s]\zeta[t] = \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{t-1}\binom{t-1}{b}(1-q)^b\,\zeta[t+a,s-a-b]$$
$$+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\,\zeta[s+a,t-a-b]$$
$$-\sum_{j=1}^{\min(s,t)}\frac{(s+t-j-1)!}{(s-j)!\,(t-j)!}\cdot\frac{(1-q)^j}{(j-1)!}\,\varphi[s+t-j].$$

Observe that the limiting case $q=1$ of Theorem 14 reduces to Euler's decomposition formula

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1}\binom{a+t-1}{t-1}\zeta(t+a,s-a) + \sum_{a=0}^{t-1}\binom{a+s-1}{s-1}\zeta(s+a,t-a).$$

A Differential Identity

The proof of Theorem 14 relies on the following identity.

Lemma 15 *Let s and t be positive integers, and let x and y be non-zero real numbers. Then for all real q such that $x + y + (q-1)xy \neq 0$,*

$$\frac{1}{x^s y^t}$$

$$= \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} \frac{(1-q)^b (1+(q-1)y)^a (1+(q-1)x)^{t-1-b}}{x^{s-a-b}(x+y+(q-1)xy)^{t+a}}$$

$$+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} \frac{(1-q)^b (1+(q-1)x)^a (1+(q-1)y)^{s-1-b}}{y^{t-a-b}(x+y+(q-1)xy)^{s+a}}$$

$$- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!\,(t-j)!} \cdot \frac{(1-q)^j}{(j-1)!} \cdot \frac{(1+(q-1)y)^{s-j}(1+(q-1)x)^{t-j}}{(x+y+(q-1)xy)^{s+t-j}}.$$

Proof of the Differential Identity

Apply the partial differential operator

$$\frac{1}{(s-1)!}\left(-\frac{\partial}{\partial x}\right)^{s-1}\frac{1}{(t-1)!}\left(-\frac{\partial}{\partial y}\right)^{t-1}$$

to both sides of the identity

$$\frac{1}{xy} = \frac{1}{x+y+(q-1)xy}\left(\frac{1}{x}+\frac{1}{y}+q-1\right).$$

$\square$

Observe that when $q = 1$, Lemma 15 reduces to the identity

$$\frac{1}{x^s y^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a}(x+y)^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{(x+y)^{s+a} y^{t-a}},$$

from which Euler's decomposition formula for $\zeta(s)\zeta(t)$ follows immediately on summing over all positive integers x and y.

The more general q-decomposition formula for $\zeta[s]\zeta[t]$ is obtained by first putting $x = [u]_q$ and $y = [v]_q$ in Lemma 15 and summing over all positive integers u and v.

Note that if $x = [u]_q$ and $y = [v]_q$, then $1+(q-1)x = q^u$, $1+(q-1)y = q^v$ and

$$[u+v]_q = x + y + (q-1)xy = [u]_q + [v]_q + (q-1)[u]_q[v]_q.$$

q-analog of Euler's reduction formula

Introduce additional q-analogs of $\zeta(s,t)$ by defining

$$\zeta_1[s,t] := (-1)^t \sum_{u>v>0}^{\infty} \frac{q^{(s-1)u+(t-1)(-v)}}{[u]_q^s[-v]_q^t} = \sum_{u>v>0}^{\infty} \frac{q^{(s-1)u+v}}{[u]_q^s[v]_q^t}$$

and

$$\zeta_2[s,t] := (-1)^s \sum_{u>v>0}^{\infty} \frac{q^{(s-1)(-u)+(t-1)v}}{[-u]_q^s[v]_q^t} = \sum_{u>v>0}^{\infty} \frac{q^{u+(t-1)v}}{[u]_q^s[v]_q^t}.$$

Let

$$\zeta_-[s] := \sum_{n=1}^{\infty} \frac{q^{(s-1)(-n)}}{[-n]_q^s} = (-1)^s \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s}$$

and for convenience, put

$$\zeta_\pm[s] := \zeta[s] + \zeta_-[s] = \sum_{0\neq n\in\mathbf{Z}} \frac{q^{(s-1)n}}{[n]_q^s}.$$

As defined previously, let

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s].$$

We also employ the notation

$$\binom{z}{a,\ b} := \binom{z}{a}\binom{z-a}{b} = \binom{z}{b}\binom{z-b}{a} = \binom{z}{a+b}\frac{(a+b)!}{a!\,b!}$$

for the trinomial coefficient, in which a, b are nonnegative integers, and which reduces to $z!/a!\,b!(z-a-b)!$ if z is an integer not less than $a+b$.

Theorem 16 (q-analog of Euler's double zeta reduction) *Let $s > 1$ and $t > 1$ be integers, and let $0 < q < 1$. Then*

$$
\begin{aligned}
&(-1)^t\zeta_1[s,t] - (-1)^s\zeta_2[s,t] \\
&\quad = \sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}(1-q)^b\Big(\zeta_{\pm}[s-a-b]\zeta[a+t] \\
&\qquad\qquad - \zeta[s+t-b] - (1-q)\zeta[s+t-b-1]\Big) \\
&\quad + \sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}(1-q)^b\Big(\zeta_{\pm}[t-a-b]\zeta[a+s] \\
&\qquad\qquad - \zeta[s+t-b] - (1-q)\zeta[s+t-b-1]\Big) \\
&\quad - \sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^{j-1}\Big(2\zeta[s+t-j+1] - (1-q)\varphi[s+t-j]\Big) \\
&\quad - \zeta_{\pm}[s]\zeta[t] + (-1)^s\sum_{k=0}^{s-1}\binom{s-1}{k}(1-q)^k\zeta[s+t-k].
\end{aligned}
$$

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