Hypergeometric Functions that Generate Series Acceleration Formulae for Values of the Riemann Zeta Function

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The Riemann zeta function is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \qquad \Re(s) > 1.$$

The central binomial coefficient is

$$\binom{2k}{k} = \frac{(2k)!}{k!\,k!} = \prod_{j=1}^{k} \frac{k+j}{j}, \qquad 1 \le k \in \mathbf{Z}.$$

A. A. Markoff (c. 1890):

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4\binom{2k}{k}}$$

Conjecture 1

$$\zeta(5) \Big/ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} \quad \text{and} \quad \zeta(6) \Big/ \sum_{k=1}^{\infty} \frac{1}{k^6\binom{2k}{k}}$$

are both transcendental.

Neither are zeros of polynomials of low degree with reasonably sized integer coefficients.

Theorem 1 *Let*

$$p(x) = a_0 + a_1 x + \cdots + a_{25}x^{25} = \sum_{j=0}^{25} a_j x^j \in \mathbf{Z}[x]$$

be a polynomial of degree at most 25*, with integer coefficients. Let*

$$x_0 = \zeta(5) \Big/ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}}.$$

If $p(x_0) = 0$*, then*

$$a_0^2 + a_1^2 + \cdots + a_{25}^2 = \sum_{j=0}^{25} a_j^2 > 10^{766}.$$

M. Koecher (1980):

$$\zeta(5) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \sum_{j=1}^{k-1}\frac{1}{j^2},$$

$$\zeta(7) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} \left\{\frac{1}{k^2} - \sum_{j=1}^{k-1}\frac{1}{j^2}\right\}$$

$$+ \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \sum_{m=1}^{k-1}\frac{1}{m^2}\sum_{j=1}^{m-1}\frac{1}{j^2},$$

and more generally, for complex z not a non-zero integer,

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1 - z^2/k^2)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \left(\frac{1}{2} + \frac{2}{1 - z^2/k^2}\right) \prod_{m=1}^{k-1}\left(1 - \frac{z^2}{m^2}\right).$$

Expanding both sides of Koecher's identity

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-z^2/k^2)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\left(\frac{1}{2}+\frac{2}{1-z^2/k^2}\right)\prod_{m=1}^{k-1}\left(1-\frac{z^2}{m^2}\right).$$

in powers of z^2 and comparing coefficients of z^{2n} for $0 \le n \in \mathbf{Z}$ gives

$$\zeta(2n+3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}(-1)^n e_n(k) + 2\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2j+3}\binom{2k}{k}}(-1)^{n-j}e_{n-j}(k),$$

where for positive integers r,

$$e_r(k) := \sum_{1\le j_1<j_2<\cdots<j_r\le k-1} \frac{1}{(j_1 j_2 \cdots j_r)^2},$$

$$e_0(k) := 1.$$

Koecher's formula

$$\zeta(5) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\left\{\frac{2}{k^2}-\frac{5}{2}\sum_{j=1}^{k-1}\frac{1}{j^2}\right\}$$

suggests seeking rational numbers a, b, c such that

$$\zeta(7) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\left\{\frac{a}{k^4}+\frac{b}{k^2}\sum_{j=1}^{k-1}\frac{1}{j^2}+c\sum_{j=1}^{k-1}\frac{1}{j^4}\right\}.$$

B. (1996): $a = 5/2$, $b = 0$, $c = 25/2$ works. Explicitly,

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}.$$

This is simpler than Koecher's formula for $\zeta(7)$ in that it involves fewer summations.

Extensive computer searches indicate that it's likely the simplest formula of its kind.

J. Borwein, B. (1996): For complex z, except for the obvious poles,

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-z^4/k^4)} = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}(1-z^4/k^4)}\prod_{m=1}^{k-1}\frac{1+4z^4/m^4}{1-z^4/m^4}.$$

Expanding both sides in powers of z^4 and comparing coefficients of z^{4n} gives a formula for $\zeta(4n+3)$ for each non-negative integer n.

In particular, comparing constant terms ($n = 0$) recovers Markoff's formula for $\zeta(3)$.

Similarly, comparing coefficients of z^4 recovers the aforementioned

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\left\{1+5k^4\sum_{j=1}^{k-1}\frac{1}{j^4}\right\}.$$

Theorem 2 (B. 2002) *Let b and c be complex numbers. Apart from the obvious poles, we have*

$$\sum_{k=1}^{\infty} \frac{k}{k^4+bk^2+c} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k\binom{2k}{k}}\cdot\frac{5k^2+b}{k^4+bk^2+c}\prod_{j=1}^{k-1}\frac{j^4+2bj^2+b^2-4c}{j^4+bj^2+c}.$$

Setting $b = -z^2$ and $c = 0$ yields Koecher's identity.

Setting $b = 0$ and $c = -z^4$ yields the Borwein-B identity.

Theorem 2 was originally a conjecture of H. Cohen.

Unaware that Theorem 2 had been proved in 2002, T. Rivoal gave a similar proof in 2006.

Hypergeometric Reformulations

The rising factorial (Pochhammer symbol) is defined for complex a by

$$(a)_r = a(a+1)\cdots(a+r-1) = \prod_{j=0}^{r-1}(a+j)$$

if r is a positive integer, and

$$(a)_0 := 1.$$

For non-negative integers p and q, the generalized hypergeometric series

$${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix}\,\middle|\, z\right) := \sum_{r=0}^{\infty}\frac{z^r\prod_{j=1}^{p}(a_j)_r}{r!\prod_{j=1}^{q}(b_j)_r}$$

is defined for complex z with $|z|<1$ if the parameters a_j and b_j are complex numbers with no b_j equal to a non-positive integer.

It often occurs that $p = q+1$. In this case, the series converges also when $|z| = 1$ if

$$\sum_{j=1}^{q}\Re(b_j) > \sum_{j=1}^{p}\Re(a_j).$$

The equivalent hypergeometric formulation of Koecher's generating function identity is

$${}_4F_3\left(\begin{matrix} 1,\ 1,\ 1+z,\ 1-z \\ 2,\ 2+z,\ 2-z\end{matrix}\,\middle|\, 1\right)$$

$$= {}_4F_3\left(\begin{matrix} 1+z,\ 1-z,\ 1+z,\ 1-z \\ 2+z,\ 2-z,\ 3/2\end{matrix}\,\middle|\, -\frac{1}{4}\right)$$

$$+\left(\frac{1-z^2}{4}\right){}_4F_3\left(\begin{matrix} 1,\ 1,\ 1+z,\ 1-z \\ 2,\ 2,\ 3/2\end{matrix}\,\middle|\, -\frac{1}{4}\right).$$

The equivalent hypergeometric formulation of the Borwein-B generating function identity is

$${}_5F_4\left(\begin{matrix} 2,\ 1+z,\ 1-z,\ 1+iz,\ 1-iz \\ 2+z,\ 2-z,\ 2+iz,\ 2-iz\end{matrix}\,\middle|\, 1\right)$$

$$= \left(\frac{5}{4}\right){}_6F_5\left(\begin{matrix} 2,\ 2,\ 1\pm z\pm iz \\ 2\pm z,\ 2\pm iz,\ 3/2\end{matrix}\,\middle|\, -\frac{1}{4}\right).$$

The equivalent hypergeometric formulation of the Cohen-B generating function identity is

$$\left(\frac{4}{5+b}\right){}_5F_4\left(\begin{matrix} 2,\ 1\pm s,\ 1\pm t \\ 2\pm s,\ 2\pm t\end{matrix}\,\middle|\, 1\right)$$

$$= {}_8F_7\left(\begin{matrix} 1,\ 1,\ 2\pm i\sqrt{b/5},\ 1\pm\sqrt{-b\pm 2\sqrt{c}} \\ 3/2,\ 1\pm i\sqrt{b/5},\ 2\pm s,\ 2\pm t\end{matrix}\,\middle|\, -\frac{1}{4}\right),$$

where

$$s = \sqrt{-\tfrac{1}{2}b + \tfrac{1}{2}\sqrt{b^2-4c}},$$

$$t = \sqrt{-\tfrac{1}{2}b - \tfrac{1}{2}\sqrt{b^2-4c}}.$$

B. (2002): Let n be a positive integer, and let b be a complex number. Apart from the poles at $b = -n^2 - j^2$, $n+1 \le j \le 2n$, we have

$${}_7F_6\left(\begin{matrix} n,\ n+1\pm i\sqrt{b/5},\ n\pm\sqrt{-b\pm 2in\sqrt{n^2+b}} \\ n+1/2, 2n+1, n\pm i\sqrt{b/5},\ n+1\pm i\sqrt{n^2+b}\end{matrix}\,\middle|\, -\frac{1}{4}\right) = \prod_{j=n+1}^{2n}\left(1+\frac{n^2+b}{j^2}\right)^{-1}.$$

B. (2002): Let n be a positive integer, and let c be a complex number. Apart from the poles at $c = j^2n^2$, $n+1 \le j \le 2n$, we have

$${}_7F_6\left(\begin{matrix} n,\ n+1\pm\sqrt{(n^4+c)/5n^2},\ 2n\pm\sqrt{c}/n,\ \pm\sqrt{c}/n \\ n+1/2,\ n\pm\sqrt{(n^4+c)/5n^2},\ 2n+1,\ n+1\pm\sqrt{c}/n\end{matrix}\,\middle|\, -\frac{1}{4}\right) = \prod_{j=n+1}^{2n}\left(1-\frac{c}{j^2n^2}\right)^{-1}.$$

M. Koecher (1980):

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2-z^2)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\left(\frac{2}{1-z^2/k^2}+\frac{1}{2}\right)\prod_{m=1}^{k-1}\left(1-\frac{z^2}{m^2}\right)$$

generates the values $\zeta(2n+3)$ for $0 \le n \in \mathbf{Z}$.

D. Leshchiner (1981):

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2-z^2} = \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}\left(\frac{2}{1-z^2/k^2}-\frac{1}{2}\right)\prod_{m=1}^{k-1}\left(1-\frac{z^2}{m^2}\right)$$

generates the values $\zeta(2n+2)$ for $0 \le n \in \mathbf{Z}$.

J. Borwein, B. (1996):

$$\sum_{k=1}^{\infty} \frac{k}{z^4-k^4} = \frac{5}{2}\sum_{k=1}^{\infty} \frac{k}{\binom{2k}{k}(z^4-k^4)}\prod_{m=1}^{k-1}\frac{4z^4+m^4}{z^4-m^4}.$$

generates the values $\zeta(4n+3)$ for $0 \le n \in \mathbf{Z}$.

D. Bailey, J. Borwein, B. (2005):

$$\sum_{k=1}^{\infty} \frac{1}{z^2-k^2} = 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(z^2-k^2)}\prod_{m=1}^{k-1}\frac{4z^2-m^2}{z^2-m^2}$$

generates the values $\zeta(2n+2)$ for $0 \le n \in \mathbf{Z}$.

The latter has equivalent hypergeometric formulation

$${}_3F_2\left(\begin{matrix} 1, & 1+z, & 1-z \\ & 2+z, & 2-z \end{matrix}\middle| 1\right) = \left(\frac{3}{2}\right){}_4F_3\left(\begin{matrix} 1, & 2, & 1+2z, & 1-2z \\ & 3/2, & 2+z, & 2-z \end{matrix}\middle| \frac{1}{4}\right).$$

Proof of the BBB Identity

By partial fractions,

$$\frac{3}{z^2-k^2}\prod_{m=1}^{k-1}\frac{4z^2-m^2}{z^2-m^2} = \sum_{n=1}^{k}\frac{c_n(k)}{z^2-n^2},$$

where

$$c_n(k) = 3\prod_{m=1}^{k-1}(4n^2-m^2) \Big/ \prod_{\substack{m=1\\ m\neq n}}^{k}(n^2-m^2)$$

if $1 \le n \le$k, and $c_n(k) = 0$ if $n > k$ or if $k \ge 2n+1$.

Therefore,

$$3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(z^2-k^2)}\prod_{m=1}^{k-1}\frac{4z^2-m^2}{z^2-m^2} = \sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}\sum_{n=1}^{k}\frac{c_n(k)}{z^2-n^2} = \sum_{n=1}^{\infty}\frac{1}{z^2-n^2}\sum_{k=n}^{2n}\frac{c_n(k)}{\binom{2k}{k}}.$$

It now suffices to show that

$$S_n := \sum_{k=n}^{2n}\frac{c_n(k)}{\binom{2k}{k}} = 1, \qquad 0 \le n \in \mathbf{Z}.$$

But

$$S_n = \frac{6n^2}{(2n)!}\sum_{k=n}^{2n}\frac{\prod_{m=1}^{k-1}(4n^2-m^2)}{\binom{2k}{k}\prod_{m=n+1}^{k}(n^2-m^2)}$$

$$= \frac{(3n)!\,n!}{(2n)!\,(2n)!}\,{}_3F_2\left(\begin{matrix} 3n, & n+1, & -n \\ 2n+1, & n+1/2 & \end{matrix}\middle| \frac{1}{4}\right).$$

Clearly $S_0 = 1$. The MAPLE command "simplify" applied to the symbolic ratio S_{n+1}/S_n yields 1.

That is, for all non-negative integers n, $S_{n+1}/S_n = 1$.

By induction, it follows that $S_n = 1$ for all $n \ge 0$. □

If a certificate is desired, we can employ the Wilf-Zeilberger algorithm. In MAPLE 9.5 we set

$$r := \frac{\binom{2n}{n}}{\binom{3n}{n}}, \quad f := \frac{(3n)_k\,(n+1)_k\,(-n)_k}{(2n+1)_k\,(n+1/2)_k} \cdot \frac{(1/4)^k}{k!},$$

where the Pochhammer symbol

$$(a)_k := \prod_{j=1}^{k} (a+j-1) = a(a+1)\cdots(a+k-1).$$

Now execute:

```
> with(SumTools[Hypergeometric]):
> WZMethod(f,r,n,k,'certify'): certify;
```

which returns the certificate

```
     /     2                            \
     \11 n  + 1 + 6 n + k + 5 k n/ k
 -   -------------------------------
      3 (n - k + 1) (2 n + k + 1) n
```

This proves that for $0 \le n \in \mathbf{Z}$,

$$\sum_{k=0}^{\infty} f(n,k) = r(n).$$

Indeed, the (suppressed) output of 'WZMethod' is the WZ-pair (F,G) such that for all $n,k \ge 0$,

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k),$$

where $F(n,k) := f(n,k)/r(n)$ and the certificate is

$$R(n,k) = \frac{G(n,k)}{F(n,k)} = -\frac{(11n^2+(5k+6)n+k+1)k}{3(n-k+1)(2n+k+1)n}.$$

Since $f(n,k)=0$ if $k>n$ and $R(n,0)=0$, we see that $G(n,k) = F(n,k)R(n,k) = 0$ if $k=0$ or if $k>n$.

Therefore,

$$\sum_{k=0}^{\infty} [F(n+1,k) - F(n,k)] = \sum_{k=0}^{\infty} [G(n,k+1) - G(n,k)] = 0,$$

and thus $\sum_{k \ge 0} F(n,k)$ is independent of n.

Thus, for all $n \ge 0$,

$$\sum_{k=0}^{\infty} \frac{f(n,k)}{r(n)} = \sum_{k=0}^{\infty} F(n,k) = \sum_{k=0}^{\infty} F(0,k) = 1.$$