Hypergeometric Functions that Generate Series Acceleration Formulae for Values of the Riemann Zeta Function

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The series acceleration formula

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{n^3(2n)!}$$

was first obtained by A. Markov in 1890, but became more widely known after its appearance in connection with Apéry's proof of the irrationality of $\zeta(3)$. We outline here the role that hypergeometric functions play in generating more general series acceleration formulae for values of the Riemann zeta function at the positive integers. At odd positive integers, we have the formulae of Koecher, and later of Borwein and Bradley [1, 3]. Shortly thereafter, Henri Cohen conjectured a generalization which was later proved by Bradley [4] and subsequently and independently by Rivoal [5]. More recently, a new generating function identity yielding an analogous acceleration formula for $\zeta(2n)$ (where n is an arbitrary positive integer) was found by Bailey, Borwein and Bradley [2]. We indicate here the motivation for its discovery and the role of hypergeometric functions in its proof.

[1] Almkvist, G. and Granville, A. J., Borwein and Bradley's Ap'ery-like formulae for $\zeta(4n+3)$, *Experiment. Math.* **8** (1999), no. 2, 197–203.

[2] Bailey, D. H., Borwein, J. M., and Bradley, D. M., Experimental Determination of Apéry-like Identities for $\zeta(2n+2)$, *Experiment. Math.*, to appear.

[3] Borwein, J. M. and Bradley, D. M., Empirically determined Apéry-like formulae for $\zeta(4n+3)$, *Experiment. Math.*, **6** (1997), no. 3, 181–194.

[4] Bradley, D. M., More Apéry-like formulae: On representing values of the Riemann zeta function by infinite series damped by central binomial coefficients, *preprint* August 1, 2002. `http://www.umemat.maine.edu/faculty/bradley/papers/bivar5.pdf`

[5] Rivoal, T., Simultaneous generation of Koecher and Almkvist-Granville's Apéry-like formulae, *Experiment. Math.*, **13** (2004), 503–508.