A Combinatorial Expression from Oceanography

Problem 96-16, *by* GLENN R. IERLEY (Scripps Institute of Oceanography, University of California, San Diego) *and* OTTO G. RUEHR (Michigan Technological University).

Define

$$S_n(p) = \sum_{j=0}^{n} \left\{ \binom{pn+p+1}{pj+p-1} - \binom{pn+p+1}{pj+p-2} \right\}$$

for integers $n \geq 0$ and $p \geq 1$.

Evaluate $S_n(p)$ for $p = 1, 2, 3, 4, 5, 6$. Show how to determine a closed from for arbitrary $p \geq 1$. The case $p = 3$ occurred in an oceanographic problem modelling the Gulf stream [1].

REFERENCE

1. G. R. IERLEY AND O. G. RUEHR, *Analytic and numerical solutions of a nonlinear boundary layer problem*, Stud. Appl. Math. **75** (1986), pp. 1–36.

Solution by DAVID M. BRADLEY (Simon Fraser University, Burnaby, B.C. Canada).

Let $T_n(p) := S_{n-1}(p)$ for all positive integers n and p. Then

$$T_n(p) = \sum_{j=1}^{n} \left\{ \binom{pn+1}{pj-1} - \binom{pn+1}{pj-2} \right\}.$$

We have the telescoping sum

$$T_n(1) = \sum_{j=1}^{n} \left\{ \binom{n+1}{j-1} - \binom{n+1}{j-2} \right\} = \binom{n+1}{n-1} = \tfrac{1}{2}n(n+1), \tag{1}$$

and the alternating sum

$$\begin{aligned} T_n(2) &= \sum_{j=1}^{n} \left\{ \binom{2n+1}{2j-1} - \binom{2n+1}{2j-2} \right\} \\ &= \sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} + \binom{2n+1}{2n} - 1 \\ &= -(1-x)^{2n+1}\Big|_{x=1} + (2n+1) - 1 \\ &= 2n. \end{aligned} \tag{2}$$

For $p \geq 3$, let

$$U_n(p) := \sum_{j \equiv -1 (p)} \binom{pn+1}{j} - \sum_{j \equiv -2 (p)} \binom{pn+1}{j},$$

where the sums are over all non-negative integers j congruent to -1 modulo p, and -2 modulo p, respectively. Then $T_n(3) = 1 + U_n(3)$ and $T_n(p) = U_n(p)$ for $p > 3$. We require (see eg. I. P. GOULDEN AND D. M. JACKSON, *Combinatorial Enumeration*, John Wiley & Sons, 1983, p.13) the following

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Lemma (Series Multisection Formula). *Let $f(x) := \sum_{j\geq 0} a_j x^j$ and let $\omega := e^{2\pi i/p}$ be a primitive pth root of unity. Then for any integer d, we have*

$$\sum_{j\equiv d(p)} a_j x^j = \frac{1}{p}\sum_{k=0}^{p-1} \omega^{-kd} f(x\omega^k).$$

If we take ω as in the lemma and

$$f(x) = \sum_{j\geq 0} \binom{pn+1}{j} x^j = (1+x)^{pn+1},$$

then

$$\begin{aligned}
U_n(p) &= \frac{1}{p}\sum_{k=0}^{p-1} \omega^k(1+\omega^k)^{pn+1} - \frac{1}{p}\sum_{k=0}^{p-1} \omega^{2k}(1+\omega^k)^{pn+1} \\
&= \frac{1}{p}\sum_{k=1}^{p-1} \omega^k(1-\omega^{2k})(1+\omega^k)^{pn} \\
&= \frac{1}{p}\sum_{k=1}^{p-1} \omega^{2k}(\omega^{-k}-\omega^k)\omega^{pnk/2}(\omega^{k/2}+\omega^{-k/2})^{pn} \\
&= \frac{1}{p}\sum_{k=1}^{p-1} \omega^{2k}(-2i\sin 2\pi k/p)(-1)^{nk}(2\cos \pi k/p)^{pn} \\
&= \frac{1}{p}\sum_{k=1}^{(p-1)/2} \Big\{ \omega^{2k}(-2i\sin 2\pi k/p)(-1)^{nk}(2\cos \pi k/p)^{pn} + \omega^{2(p-k)}(-2i\sin 2\pi (p-k)/p) \\
&\qquad \times (-1)^{n(p-k)}(2\cos \pi (p-k)/p)^{pn} \Big\}.
\end{aligned}$$

When p is even, the latter sum has a vanishing $k = p/2$ term, which we have omitted. Thus,

$$\begin{aligned}
U_n(p) &= \frac{1}{p}\sum_{k=1}^{(p-1)/2} (-1)^{nk}(-2i\sin 2\pi k/p)(2\cos \pi k/p)^{pn}(\omega^{2k}-\omega^{-2k}) \\
&= \frac{1}{p}\sum_{k=1}^{(p-1)/2} (-1)^{nk}(2\sin 2\pi k/p)(2\sin 4\pi k/p)(2\cos \pi k/p)^{np}. \qquad (3)
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_n(3) &= 1 + U_n(3) \\
&= 1 + \tfrac{4}{3}(-1)^n(\sin 2\pi/3)(2\cos \pi/3)^{3n}(\sin 4\pi/3) \\
&= 1 + (-1)^{n+1}. \qquad (4)
\end{aligned}$$

For $p > 3$, we have $T_n(p) = U_n(p)$, and hence in particular,

$$T_n(4) = 0, \tag{5}$$

$$T_n(5) = \frac{(-1)^n}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{5n} - \left(\frac{1-\sqrt{5}}{2}\right)^{5n}\right\}, \tag{6}$$

$$T_n(6) = \frac{(-3)^{3n}-1}{2}. \tag{7}$$

Since $S_n(p) = T_{n+1}(p)$, we have, in view of equations (1)–(7),

$$\begin{aligned}
S_n(1) &= \tfrac{1}{2}(n+1)(n+2),\\
S_n(2) &= 2(n+1),\\
S_n(3) &= 1+(-1)^n,\\
S_n(4) &= 0,\\
S_n(5) &= \frac{(-1)^n}{\sqrt{5}}\left\{\left(\frac{1-\sqrt{5}}{2}\right)^{5n+5} - \left(\frac{1+\sqrt{5}}{2}\right)^{5n+5}\right\},\\
S_n(6) &= \frac{(-3)^{3n+3}-1}{2},
\end{aligned}$$

and in general for $p > 3$,

$$S_n(p) = U_{n+1}(p) = \frac{1}{p}\sum_{k=1}^{(p-1)/2} (-1)^{(n+1)k}(2\sin 2\pi k/p)(2\sin 4\pi k/p)(2\cos \pi k/p)^{(n+1)p}.$$

Remark. One can show directly that $T_n(4)$ (and hence $S_n(4)$) vanishes, without recourse to the formula (3) for $U_n(p)$. We have

$$\begin{aligned}
T_n(4) &= \sum_{j=1}^{n}\binom{4n+1}{4j-1} - \sum_{j=1}^{n}\binom{4n+1}{4j-2}\\
&= \sum_{j=1}^{n}\binom{4n+1}{4j-1} - \sum_{j=1}^{n}\binom{4n+1}{(4n+1)-(4j-2)}.
\end{aligned}$$

In the final sum, replace the summation variable j by $k = n - j + 1$. Then

$$T_n(4) = \sum_{j=1}^{n}\binom{4n+1}{4j-1} - \sum_{k=1}^{n}\binom{4n+1}{4k-1} = 0.$$

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