

Searching Symbolically for Apéry-like Formulae for Values of The Riemann Zeta Function

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Abstract. We discuss some aspects of the search for identities using computer algebra and symbolic methods. To keep the discussion as concrete as possible, we shall focus on so-called Apéry-like formulae for special values of the Riemann Zeta function. Many of these results are apparently new, and much more work needs to be done before they can be formally proved and properly classified. A first step in this direction can be found in [1].

1. Introduction

The Riemann Zeta function is

$$(1.1) \quad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 1.$$

In view of the “Apéry-like” formulae

$$(1.2) \quad \zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}, \quad \zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}, \quad \zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}},$$

one is tempted to speculate that there is an analogous formula for $\zeta(5)$, $\zeta(6)$, $\zeta(7)$ and so on. The key word here is *analogous*. For example, extensive computation has ruled out the possibility of formulae of the form

$$\zeta(5) = \frac{a}{b} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}}, \quad \zeta(6) = \frac{c}{d} \sum_{k=1}^{\infty} \frac{1}{k^6 \binom{2k}{k}},$$

where a , b , c , d are moderately sized integers. Such negative results are useful, as they tell us it would be a waste of time to search for interesting formulae

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of a given form. Thus, it would seem there are no corresponding Apéry-like formulae for higher zeta values. End of story. Consider however, the following result of Koecher [2, 3]:

$$(1.3) \quad \zeta(5) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2}.$$

Koecher's formula points up a potential problem with symbolic searching. Namely, negative results need to be interpreted carefully, lest they be given more weight than they deserve and unnecessarily discourage further investigation. Also, it becomes clear that symbolic searching is very much limited by the need to know fairly precisely the form of what one is searching for in advance.

Koecher's formula (1.3) suggests that one might profit by searching for a formula of the form

$$\zeta(7) = r_1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} + r_2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2} + r_3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4},$$

where r_1, r_2, r_3 are rational numbers. The following (conjectured)¹ formula for $\zeta(7)$ was found [1] using high precision arithmetic and Maple's integer relations algorithms:

$$(1.4) \quad \zeta(7) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} + \frac{25}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4}.$$

More generally, we have the (conjectured)² generating function formula [1]

$$(1.5) \quad \sum_{k=1}^{\infty} \frac{1}{k^3 (1 - z^4/k^4)} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \frac{1}{1 - z^4/k^4} \prod_{j=1}^{k-1} \frac{j^4 + 4z^4}{j^4 - z^4}, \quad z \in \mathbf{C}.$$

Note that the constant coefficient in (1.5) gives the formula for $\zeta(3)$ in (1.2). The coefficient of z^4 in (1.5) gives (1.4). We arrived at (1.5) by extensive use of Maple's lattice algorithms, combined with a good deal of insightful guessing. Interestingly, Maple's `convert(series, ratpoly)` feature played a significant role. The reader is referred to [1] for details.

Comparing our generating function formula (1.5) with Koecher's [3]

$$(1.6) \quad \sum_{k=1}^{\infty} \frac{1}{k^3 (1 - z^2/k^2)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \left(\frac{1}{2} + \frac{2}{1 - z^2/k^2} \right) \prod_{j=1}^{k-1} (1 - z^2/j^2)$$

¹See note below.

²These conjectures have subsequently been proved. See Granville and Almqvist's preprint <http://www.math.uga.edu/~andrew/Postscript/BorBrad.ps>.

raises some interesting issues related to formula redundancy, and which remain unresolved. We address certain of these issues in the next section.

2. Redundancy Relations

To mitigate the problem of symbol clutter in what follows requires some notation. We denote the power sum symmetric functions by

$$P_r(k) := \begin{cases} \sum_{j=1}^{k-1} j^{-r}, & r \geq 1, \\ 1, & r = 0. \end{cases}$$

Next, we define functions λ , μ by

$$\begin{aligned} \lambda(m, \prod_{j=1}^n P_{r_j}) &:= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^m \binom{2k}{k}} \prod_{j=1}^n P_{r_j}(k), \\ \mu(m, \prod_{j=1}^n P_{r_j}) &:= \sum_{k=1}^{\infty} \frac{1}{k^m \binom{2k}{k}} \prod_{j=1}^n P_{r_j}(k). \end{aligned}$$

In the new notation, (1.2) becomes

$$(2.1) \quad \zeta(2) = 3\mu(2, P_0), \quad \zeta(3) = \frac{5}{2}\lambda(3, P_0), \quad \zeta(4) = \frac{36}{17}\mu(4, P_0),$$

while (1.3) and (1.4) become

$$(2.2) \quad \zeta(5) = 2\lambda(5, P_0) - \frac{5}{2}\lambda(3, P_2), \quad \zeta(7) = \frac{5}{2}\lambda(7, P_0) + \frac{25}{2}\lambda(3, P_4),$$

respectively. To illustrate the issue of formula redundancy, consider Koecher's formula for $\zeta(7)$ [3] which becomes, in our notation,

$$(2.3) \quad \zeta(7) = 2\lambda(7, P_0) - 2\lambda(5, P_2) + \frac{5}{4}\lambda(3, P_2^2) - \frac{5}{4}\lambda(3, P_4).$$

In view of the second formula in (2.2), the middle two terms of (2.3) must be redundant. Indeed, lattice-based reduction shows that

$$(2.4) \quad -2\lambda(5, P_2) + \frac{5}{4}\lambda(3, P_2^2) = \frac{55}{4}\lambda(3, P_4) + \frac{1}{2}\lambda(7, P_0).$$

Although we currently have no real understanding why interrelations between λ sums such as (2.4) hold, we decided to limit our symbolic search for Zeta

function identities in which no such interrelations exist.³ This was carried out by starting with a “full set” of λ sums and checking that a relation holds with the relevant Zeta value. Now recurse, using the following scheme. From any found relation, toss out the Zeta value. If no relation is found amongst the remaining sums, output the relation that held when the Zeta value was included, and report it as non-redundant. Otherwise, systematically discard the various λ sums from the list, until a non-redundant relation remains. Carrying out the aforementioned procedure yields the following formulae which evidently exhaust the list of non-redundant formulae for each given Zeta value:

$$17\zeta(4) - 36\mu(4, P_0) = 5\zeta(4) - 108\mu(2, P_2) = 0,$$

$$\begin{aligned} &7\zeta(6) + 1944\mu(2, P_4) - 1944\mu(2, P_2^2) \\ &= 215\zeta(6) - 2592\mu(4, P_2) - 3888\mu(2, P_4) \\ &= 229\zeta(6) - 2592\mu(4, P_2) - 3888\mu(2, P_2^2) \\ &= 1481\zeta(6) - 2592\mu(6, P_0) - 3888\mu(2, P_2^2) \\ &= 313\zeta(6) - 648\mu(6, P_0) + 648\mu(4, P_2) \\ &= 163\zeta(6) - 288\mu(6, P_0) - 432\mu(2, P_4) \\ &= 0, \end{aligned}$$

$$\begin{aligned} &2\zeta(7) - 5\lambda(7, P_0) - 25\lambda(3, P_4) \\ &= 4\zeta(7) - 25\lambda(3, P_2^2) + 40\lambda(5, P_2) + 225\lambda(3, P_4) \\ &= 22\zeta(7) - 25\lambda(3, P_2^2) + 40\lambda(5, P_2) - 45\lambda(7, P_0) \\ &= 0, \end{aligned}$$

$$\begin{aligned} &72\zeta(9) + 135\lambda(7, P_2) - 147\lambda(9, P_0) - 60\lambda(5, P_2^2) - 85\lambda(3, P_6) + 25\lambda(3, P_2^3) \\ &= 36\zeta(9) - 540\lambda(5, P_4) - 96\lambda(9, P_0) + 60\lambda(5, P_2^2) - 1130\lambda(3, P_6) \\ &\quad + 675\lambda(3, P_4 P_2) - 25\lambda(3, P_2^3) \\ &= 4\zeta(9) + 196\lambda(5, P_4) + 32\lambda(7, P_2) - 36\lambda(5, P_2^2) + 390\lambda(3, P_6) \\ &\quad - 245\lambda(3, P_4 P_2) + 15\lambda(3, P_2^3) \\ &= 4\zeta(9) - 20\lambda(5, P_4) + 5\lambda(7, P_2) - 9\lambda(9, P_0) - 45\lambda(3, P_6) + 25\lambda(3, P_4 P_2) \\ &= 116\zeta(9) + 68\lambda(5, P_4) + 226\lambda(7, P_2) - 234\lambda(9, P_0) - 108\lambda(5, P_2^2) \\ &\quad - 85\lambda(3, P_4 P_2) + 45\lambda(3, P_2^3) \\ &= 0. \end{aligned}$$

³Of course, we cannot prove that (1.4) contains no redundancy, since, for example, we cannot even prove that $\zeta(7)$ is irrational.

No additional formulae other than the formulae given in §1 were found for $\zeta(2)$, $\zeta(3)$ and $\zeta(5)$. We discuss additional uniqueness issues in the next section.

3. Uniqueness and $\zeta(4n + 3)$

If one extends the list given in the previous section, it becomes apparent that $\zeta(4n + 3)$ evidently has a unique representation in terms of λ sums of the form $\lambda(m, P_r)$ in which r is always a multiple of four. We exploited this observation in [1] to arrive at our generating function formula (1.5). Unfortunately, there seems to be no sensible selection to make amongst the formulae for $\zeta(4n + 1)$ which gives an analogous generating function identity. Our Maple code for producing all possible non-redundant formulae for $\zeta(13)$ ran for over two months before it was killed. The resulting incomplete file is over three thousand lines long and contains hundreds and hundreds of independent formulae. If a generating function identity (other than a bisection of Koecher's) for $\zeta(4n + 1)$ is found, it is unlikely that it will be discovered by hunting for the appropriate representatives from the identities for $\zeta(9)$, $\zeta(13)$, etc. and looking for a pattern.

Recall Ramanujan's formulae [4]

$$2\zeta(4n + 3) = (2\pi)^{4n+3} \sum_{k=0}^{2n+2} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{4n+4-2k}}{(4n+4-2k)!} - 4 \sum_{k=1}^{\infty} \frac{k^{-4n-3}}{e^{2\pi k} - 1}$$

and

$$2\zeta(4n + 1) = (2\pi)^{4n+1} \frac{1}{2n} \sum_{k=0}^{2n+1} (-1)^{k+1} (2k-1) \frac{B_{2k}}{(2k)!} \frac{B_{4n+2-2k}}{(4n+2-2k)!} - 4 \sum_{k=1}^{\infty} \frac{k^{-4n-1}}{e^{2\pi k} - 1} - \frac{\pi}{n} \sum_{k=1}^{\infty} \frac{k^{-4n}}{\sinh^2(\pi k)}.$$

Here, the additional complexity in the $4n + 1$ case arises from taking the derivative of the appropriate modular transformation formula. Perhaps there is an analogous phenomenon operating in the case of these Apéry-like identities as well.

References

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