ON A CLAIM OF RAMANUJAN ABOUT CERTAIN HYPERGEOMETRIC SERIES

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Abstract. We state and prove a claim of Ramanujan. As a consequence, a large new class of Saalschützian hypergeometric series is summed in closed form.

Although Ramanujan published no papers on hypergeometric series, chapters 10 and 11 of his second notebook [5] deal almost exclusively with this subject. Furthermore, several other results on hypergeometric functions are scattered among the 100 pages of unorganized material at the end of the second notebook. In 1923, Hardy [4] published a brief survey of chapter 10's corresponding chapter in Ramanujan's first notebook, namely chapter 12. There we see that Ramanujan had rediscovered many of the classical formulae of the subject, including the theorems of Gauss, Kummer, Dougall, Dixon, and Saalschütz. However, in addition, Ramanujan discovered many new theorems about hypergeometric series, in particular, theorems on products of hypergeometric series and several types of asymptotic expansions. For proofs, see Berndt [3]. In this paper, we examine an enigmatic claim about hypergeometric series made by Ramanujan in the unorganized portion of the second notebook, and it is to this claim that we now turn.

At the top of page 280 in his second notebook [5], Ramanujan states "the difference between

$$\frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)}$$

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and

$$\frac{\Gamma(\beta+1)}{\Gamma(a+\beta+1)}+\frac{\alpha m}{1!}\cdot\frac{\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+n+2)}+\frac{\alpha(\alpha+1)}{2!}\cdot\frac{m(m+2n+1)\Gamma(\beta+2n+1)}{\Gamma(\alpha+\beta+2n+3)}+\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\cdot\frac{m(m+3n+1)(m+3n+2)\Gamma(\beta+3n+1)}{\Gamma(\alpha+\beta+3n+4)}+\cdots\text{''}$$

It is not entirely clear what Ramanujan meant by this statement, nor even what values of the parameters α, β, m, n he considered. In this paper, we give perhaps the most plausible interpretation of Ramanujan's enigmatic entry. In fact, we determine conditions for which this difference equals zero.

For each integer j and complex a, z, $a_1, \dots, a_p$, $b_1, \dots, b_q$, as customary, set

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$$

and

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\middle| z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j z^j}{(b_1)_j \cdots (b_q)_j j!}\,.$$

When $p = q+1$, ${}_pF_q$ converges at $z = 1$ if $\mathrm{Re}(b_1 + \cdots + b_q) > \mathrm{Re}(a_1 + \cdots + a_p)$. See [2, p. 8]. If $p = 2$ and $q = 1$, we have Gauss's theorem; namely,

$$(1)\qquad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c) > \mathrm{Re}(a+b).$$

See [2, p. 2] for a proof.

Let α, β, m be complex numbers. Define the function S, taking values from the open complex plane into the Riemann sphere, by

$$(2)\quad S(z) = S(\alpha, \beta, m, z) = m \sum_{j=0}^{\infty} \frac{\Gamma(\beta+1+jz)\Gamma(m+j(z+1))}{\Gamma(\alpha+\beta+1+j(z+1))\Gamma(m+jz+1)} \cdot \frac{(\alpha)_j}{j!}.$$

Note that $S(z)$ coincides with Ramanujan's hypergeometric series when the variable z is replaced by n. When the parameter α is a non–positive integer, the series terminates, and in this case, we have the following theorem.

Theorem. *Let $\alpha = -k$, where k is a nonnegative integer. Then, for every complex number z,*

$$S(z) = \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)}. \tag{3}$$

Furthermore, if $z = 0$ and $\mathrm{Re}(\beta+1-m) > 0$, then (3) is valid for all complex numbers α.

Before proving the theorem, some preliminary remarks are in order. When $z = n$ is a nonnegative integer, the series (2) reduces to

$$S(n) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \sum_{j=0}^{\infty} \frac{(\beta+1)_{nj}(m)_{(n+1)j}(\alpha)_j}{(\alpha+\beta+1)_{(n+1)j}(m+1)_{nj}j!}, \tag{4}$$

which can be recast in the form

(5)

$$S(n) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$$

$$\times\, {}_{2n+2}F_{2n+1}\left(\begin{matrix} \frac{\beta+1}{n}, \frac{\beta+2}{n}, \ldots, \frac{\beta+n}{n}, \frac{m}{n+1}, \frac{m+1}{n+1}, \ldots, \frac{m+n}{n+1}, \alpha \\ \frac{m+1}{n}, \frac{m+2}{n}, \ldots, \frac{m+n}{n}, \frac{\alpha+\beta+1}{n+1}, \frac{\alpha+\beta+2}{n+1}, \ldots, \frac{\alpha+\beta+1+n}{n+1} \end{matrix} \middle| 1 \right)$$

by means of Gauss's multiplication formula. Observe that ${}_{2n+2}F_{2n+1}$ in (5) is Saalschützian for $n > 0$, i.e., the sum of the denominator parameters exceeds the sum of the numerator parameters by 1. Thus, for each positive integer n, ${}_{2n+2}F_{2n+1}$ converges for all complex α, β, m by the previously cited remarks on convergence.

R. Askey observed that in the form (5), the terminating case with $n = 1$ follows readily from an unpublished result of Askey and Ismail [1]. By utilizing Euler's evaluation of the Beta integral and a transformation of Pfaff, they showed that, if $\mathrm{Re}(d) > \mathrm{Re}(a) > 0$ and k is a nonnegative integer, then

$${}_4F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, -k, c \\ \frac{d}{2}, \frac{d+1}{2}, -k+a+c+1-d \end{matrix} \middle| 1 \right) = \frac{(d-a)_k(d-c)_k}{(d-a-c)_k(d)_k}\, {}_3F_2\left(\begin{matrix} -k, a, c \\ k+d, d-c \end{matrix} \middle| 1 \right).$$

As Askey observed, if c tends to $k+d$, the expression on the left becomes a ${}_4F_3$ of the form occurring in (5) with $n=1$. The expressions on the right are easily evaluated using the fact that

$$\lim_{\epsilon\to 0} {}_2F_1\left(\begin{matrix}-k, a\\ -k+\epsilon\end{matrix}\,\middle|\, 1\right) = \frac{(-k-a)_k}{(-k)_k}.$$

We are grateful to the referee for pointing out that the aforementioned ${}_4F_3$ transformation of Askey and Ismail is a specialization of a result due to F. J. W. Whipple, and can be obtained by replacing k by d, a by $-a+d$, c by a, then b by c and m by k in the formula (1) of [2, p. 32].

Proof of Theorem. The case $z=0$ is easily proved by setting $n=0$ in (4) and applying Gauss's theorem (1).

Next, let $\alpha=-k$. Since

$$\begin{aligned} S(z) &= m\sum_{j=0}^{k} \frac{\Gamma(\beta+1+jz)\Gamma(m+j(z+1))(-k)_j}{\Gamma(-k+\beta+1+j(z+1))\Gamma(m+jz+1)j!} \\ &= m\sum_{j=0}^{k}(-k+\beta+1+j(z+1))_{k-j}(m+jz+1)_{j-1}(-k)_j/j! \end{aligned}$$

is a polynomial in z of degree $k-1$, it suffices to prove (3) for k distinct values of z. We shall, in fact, prove that (3) holds for all positive integers z, since the argument is no more difficult than the argument for only k values of z. So let $z=n$ be a positive integer. From (4), since $\alpha=-k$, we have

(6)

$$\begin{aligned} \frac{\Gamma(\alpha+\beta+1-m)}{\Gamma(\beta+1-m)}S(n) &= \frac{\Gamma(\beta+1)\Gamma(\alpha+\beta+1-m)}{\Gamma(\alpha+\beta+1)\Gamma(\beta+1-m)}\sum_{j=0}^{k}\frac{(\beta+1)_{nj}(m)_{(n+1)j}(\alpha)_j}{(\alpha+\beta+1)_{(n+1)j}(m+1)_{nj}j!} \\ &= \frac{\Gamma(m-\beta)\Gamma(-\alpha-\beta)}{\Gamma(-\beta)\Gamma(m-\alpha-\beta)}\sum_{j=0}^{k}\frac{(\beta+1)_{nj}(m)_{(n+1)j}(\alpha)_j}{(\alpha+\beta+1)_{(n+1)j}(m+1)_{nj}j!} \end{aligned}$$

by the reflection formula for Γ. The sine factors that would normally appear reduce to unity because α is an integer. By Gauss's theorem (1), for $\mathrm{Re}(-\alpha-\beta-j(n+1)) > 0$, we have

(7)

$$
\begin{aligned}
{}_2F_1\left(\left.\begin{matrix} m+j(n+1), \alpha+j \\ m-\beta+j \end{matrix}\right| 1\right) &= \frac{\Gamma(m-\beta+j)\Gamma(-\alpha-\beta-j(n+1))}{\Gamma(m-\alpha-\beta)\Gamma(-\beta-jn)} \\
&= \frac{\Gamma(m-\beta)(m-\beta)_j\Gamma(-\alpha-\beta)(-\beta-jn)_{nj}}{\Gamma(m-\alpha-\beta)(-\alpha-\beta-j(n+1))_{(n+1)j}\Gamma(-\beta)} \\
&= \frac{\Gamma(m-\beta)\Gamma(-\alpha-\beta)}{\Gamma(-\beta)\Gamma(m-\alpha-\beta)} \cdot \frac{(m-\beta)_j(\beta+1)_{nj}(-1)^j}{(\alpha+\beta+1)_{(n+1)j}}.
\end{aligned}
$$

The condition $\mathrm{Re}(-\alpha-\beta-j(n+1)) > 0$ is certainly satisfied if $\mathrm{Re}(\beta) < -nk$. For now, assume this. Then, (6) and (7) imply that

(8)

$$
\begin{aligned}
S(n) &= \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)} \sum_{j=0}^{k} \frac{(\alpha)_j(m)_{(n+1)j}(-1)^j}{(m-\beta)_j(m+1)_{nj}j!} {}_2F_1\left(\left.\begin{matrix} m+j(n+1), \alpha+j \\ m-\beta+j \end{matrix}\right| 1\right) \\
&= \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)} \sum_{j=0}^{k} \frac{(\alpha)_j(m)_{(n+1)j}(-1)^j}{(m-\beta)_j(m+1)_{nj}j!} \sum_{s=0}^{\infty} \frac{(m+j(n+1))_s(\alpha+j)_s}{(m-\beta+j)_s s!} \\
&= \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)} \sum_{j=0}^{k} \frac{(-1)^j/j!}{(m+1)_{nj}} \sum_{s=0}^{\infty} \frac{(m)_{s+j(n+1)}(\alpha)_{s+j}}{(m-\beta)_{s+j}s!} \\
&= \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)} \sum_{j=0}^{k} \frac{(-1)^j/j!}{(m+1)_{nj}} \sum_{r=j}^{\infty} \frac{(m)_{r+nj}(\alpha)_r}{(m-\beta)_r(r-j)!} \\
&= \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)} \sum_{r=0}^{\infty} \frac{(\alpha)_r(m)_r}{(m-\beta)_r r!} \sum_{j=0}^{r} \frac{(m+r)_{nj}}{(m+1)_{nj}}(-1)^j \binom{r}{j}.
\end{aligned}
$$

Interchanging the order of summation is justified since both sums contain only finitely many nonzero terms.

Let E denote the inner sum in (8). For $r = 0$, it is clear that $E = 1$. For $r \geq 1$, we shall prove that $E = 0$. Note that

$$\frac{(m+r)_{nj}}{(m+1)_{nj}} = \frac{\Gamma(m+r+nj)\Gamma(m+1)}{\Gamma(m+1+nj)\Gamma(m+r)} = \frac{(m+nj+1)_{r-1}}{(m+1)_{r-1}}.$$

Thus,

$$E = \sum_{j=0}^{r} \frac{(m+nj+1)_{r-1}}{(m+1)_{r-1}}(-1)^j \binom{r}{j}.$$

But,

$$(k)_j = D^j x^{k+j-1}\big|_{x=1}.$$

Therefore,

$$\begin{aligned}
E &= \frac{1}{(m+1)_{r-1}} \sum_{j=0}^{r} D^{r-1} x^{(m+nj+1)+(r-1)-1} (-1)^j \binom{r}{j}\Bigg|_{x=1} \\
&= \frac{D^{r-1} x^{m+r-1}}{(m+1)_{r-1}} \sum_{j=0}^{r} x^{nj} (-1)^j \binom{r}{j}\Bigg|_{x=1} \\
&= \frac{D^{r-1} x^{m+r-1} (1-x^n)^r}{(m+1)_{r-1}}\Bigg|_{x=1} \\
&= 0,
\end{aligned}$$

as required.

Hence, for all positive integers n, if $\mathrm{Re}(\beta) < -nk$, (3) holds. Since both sides of (3) define meromorphic functions of β for fixed n, m and $\alpha = -k$, the restriction $\mathrm{Re}(\beta) < -nk$ may be removed by analytic continuation. By our earlier remarks, this completes the proof of the theorem.

In general, if $S(z)$ is nonterminating, (3) is false. For example, let α, β, m be any complex numbers satisfying $\alpha + \beta + 1 = m$. By (5),

$$\begin{aligned}
S(1) &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, {}_4F_3\left(\begin{matrix} \beta+1, \frac{m}{2}, \frac{m+1}{2}, \alpha \\ m+1, \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2} \end{matrix} \middle| 1 \right) \\
&= \frac{\Gamma(\beta+1)}{\Gamma(m)}\, {}_2F_1\left(\begin{matrix} \beta+1, \alpha \\ m+1 \end{matrix} \middle| 1 \right) \\
&= \frac{\Gamma(\beta+1)\Gamma(m+1)\Gamma(m-\beta-\alpha)}{\Gamma(m)\Gamma(m-\beta)\Gamma(m+1-\alpha)} \\
&= \frac{m}{(m-\alpha)\Gamma(\alpha+1)}.
\end{aligned} \tag{9}$$

But

$$\frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)} = \frac{\Gamma(-\alpha)}{\Gamma(0)} (= 0 \quad \text{for} \quad \alpha \neq 0, 1, 2, 3 \ldots). \tag{10}$$

When α is a non–positive integer, (9) and (10) must be equal, as we have seen. Indeed, they both vanish when α is a negative integer. However, it is clear that

(9) and (10) are unequal for non–integral α: (10) vanishes while (9) does not. We have used MAPLE to calculate several other nonterminating examples and found no instances when (3) is valid.

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