

SIGNED q -ANALOGS OF TORNHEIM'S DOUBLE SERIES

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ABSTRACT. We introduce signed q -analogs of Tornheim's double series, and evaluate them in terms of double q -Euler sums. As a consequence, we provide explicit evaluations of signed and unsigned Tornheim double series, and correct some mistakes in the literature.

1. INTRODUCTION

Let k be a positive integer. Sums of the form

$$\zeta(s_1, s_2, \dots, s_k) := \sum_{n_1 > n_2 > \dots > n_k > 0} \prod_{j=1}^k n_j^{-s_j}, \quad \sum_{j=1}^m \Re(s_j) > m, \quad m = 1, 2, \dots, k. \quad (1)$$

have attracted increasing attention in recent years; see eg. [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 16, 24, 26, 27, 29, 30, 31]. The survey articles [7, 18, 19, 36, 37, 38] provide an extensive list of references. In (1) the sum is over all positive integers n_1, n_2, \dots, n_k satisfying the indicated inequalities. Of course (1) reduces to the familiar Riemann zeta function when $k = 1$. When the arguments are all positive integers, we refer to (1) as a multiple zeta value and note that in this case, $s_1 > 1$ is necessary and sufficient for convergence.

The problem of evaluating $\zeta(s_1, s_2)$ with integers $s_1 > 1$ and $s_2 > 0$ seems to have been first proposed in a letter from Goldbach to Euler [21] in 1742. (See also [20, 22] and [1, p. 253].) Calculating several examples led Euler to infer a closed form evaluation in terms of values of the Riemann zeta function in the case when $s_1 + s_2$ is odd. In [4], Borwein, Bradley and Broadhurst considered the more general Euler sum

$$\zeta(s_1, s_2, \dots, s_k; \sigma_1, \sigma_2, \dots, \sigma_k) := \sum_{n_1 > n_2 > \dots > n_k > 0} \prod_{j=1}^k \sigma_j^{n_j} n_j^{-s_j} \quad (2)$$

with each $\sigma_j \in \{-1, 1\}$. Among the many other results for (2) listed therein is an explicit formula for the case $k = 2$ that reduces to Euler's evaluation when $\sigma_1 = \sigma_2 = 1$. For the case of arbitrary k , several infinite classes of closed form evaluations for (1) and (2) are

Date: July 18, 2008.

1991 Mathematics Subject Classification. Primary: 11M41; Secondary: 11M06, 05A30, 33E20, 30B50.

Key words and phrases. Tornheim's double series, alternating Euler sums, multiple harmonic series, multiple zeta values, q -analog, q -series.

proved in [5, 6, 8, 9, 10]. Subsequently, Bradley [12, 13, 14, 15] found q -analogs of (1) and (2) and thoroughly investigated their properties.

In [32] Tornheim introduced the double series

$$T(r, s, t) := \sum_{u, v=1}^{\infty} \frac{1}{u^r v^s (u+v)^t},$$

with nonnegative integers r, s and t satisfying $r+t > 1$, $s+t > 1$ and $r+s+t > 2$. Huard, Williams and Zhang [25] evaluated Tornheim's series in terms of sums of products of Riemann zeta values when $r+s+t$ is odd. Subbarao and Sitaramachandrarao [28] considered the alternating variants

$$R(r, s, t) := \sum_{u, v=1}^{\infty} \frac{(-1)^v}{u^r v^s (u+v)^t}, \quad S(r, s, t) := \sum_{u, v=1}^{\infty} \frac{(-1)^{u+v}}{u^r v^s (u+v)^t},$$

and posed the problem to evaluate $R(r, r, r)$ and $S(r, r, r)$ for any positive integer r . Tsumura [33, Cor. 3] and [34, Theorem 3.6] tackled this problem and later [35] evaluated $S(r, s, t)$ for any positive integers r, s, t such that $r+s+t$ is odd. Tsumura's method is elementary but complicated, and also has some mistakes. In this paper, we give a simple formula that expresses R, S and T in terms of double Euler sums. In light of the aforementioned formula of Borwein, Bradley and Broadhurst for the double Euler sums, our results yield a closed form evaluation for the series R, S and T whenever the arguments r, s and t are positive integers with $r+s+t$ odd. More generally, we consider q -analogs of R, S, T and show how they may all be evaluated in terms of double q -Euler sums.

2. Q-ANALOGS

Henceforth assume q is real and $q > 1$. The q -analog of a positive integer n is

$$[n]_q := \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1}.$$

Let k be a positive integer, let s_1, s_2, \dots, s_k be real numbers, and let $\sigma_1, \sigma_2, \dots, \sigma_k \in \{-1, 1\}$. Define the q -Euler sum

$$\zeta_q[s_1, s_2, \dots, s_k; \sigma_1, \sigma_2, \dots, \sigma_k] := \sum_{n_1 > n_2 > \dots > n_k > 0} \prod_{j=1}^k \frac{\sigma_j^{n_j} q^{(s_j-1)n_j}}{[n_j]_q^{s_j}} \quad (3)$$

and note that this coincides with the special case $\text{Li}_{s_1, \dots, s_k}[\sigma_1 q^{s_1-1}, \dots, \sigma_k q^{s_k-1}]$ of the multiple q -polylogarithm [12, eq. (6.2)]. If $\sigma_j = 1$ for each $j = 1, 2, \dots, k$, then we recover the multiple q -zeta value $\zeta[s_1, s_2, \dots, s_k]$ of [12, 14, 15].

Let $\sigma, \tau \in \{-1, 1\}$. Define q -analogs of the signed and unsigned Tornheim double series R , S and T by

$$T[r, s, t; \sigma, \tau] := \sum_{u, v=1}^{\infty} \frac{\sigma^u \tau^v q^{(r+t-1)u + (s+t-1)v}}{[u]_q^r [v]_q^s [u+v]_q^t}.$$

The sum

$$\varphi[s; \sigma] := \sum_{n=1}^{\infty} \frac{(n-1)\sigma^n q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{n\sigma^n q^{(s-1)n}}{[n]_q^s} - \zeta_q[s; \sigma],$$

the case $\sigma = 1$ of which was defined in [15], will also be needed. As in [4], it is convenient to combine signs and exponents in (2) and (3) into a single list by writing s_j if $\sigma_j = 1$ and \bar{s}_j if $\sigma_j = -1$. For consistency one may do this also for T and φ ; thus for example, $\varphi[s; 1] = \varphi[s]$, $\varphi[s; -1] = \varphi[\bar{s}]$, $R(r, s, t) = T(r, \bar{s}, t)$ and $S(r, s, t) = T(\bar{r}, \bar{s}, t)$. We also employ the notation

$$\binom{z}{a, b} := \binom{z}{a} \binom{z-a}{b} = \binom{z}{b} \binom{z-b}{a}$$

for the trinomial coefficient, in which a, b are nonnegative integers, and which reduces to $z! / a! b! (z-a-b)!$ if z is also an integer exceeding $a+b$.

The following theorem shows how the q -analogs of R , S and T are related to the q -Euler sums.

Theorem 1. *Let r and s be positive integers, and let t be a real number. Then*

$$\begin{aligned} T[r, s, t] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \zeta_q[s+t+a, r-a-b] \\ &\quad + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \zeta_q[r+t+a, s-a-b] \\ &\quad - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \varphi[r+s+t-j], \\ T[\bar{r}, \bar{s}, t] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \zeta_q[\overline{s+t+a}, r-a-b] \\ &\quad + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \zeta_q[\overline{r+t+a}, s-a-b] \\ &\quad - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \varphi[\overline{r+s+t-j}], \end{aligned}$$

$$\begin{aligned}
T[r, \bar{s}, t] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \zeta_q[\overline{s+t+a}, \overline{r-a-b}] \\
&+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \zeta_q[r+t+a, \overline{s-a-b}] \\
&- \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j (1+q)^{j-r-s-t} \zeta_{q^2}[r+s+t-j].
\end{aligned}$$

Taking the limit as $q \rightarrow 1+$ in Theorem 1 and noting the restrictions on r , s and t now needed for convergence yields the following

Corollary 1. *Let r and s be positive integers and let t be a real number. If $r+t > 1$ and $s+t > 1$, then*

$$T(r, s, t) = \sum_{a=0}^{r-1} \binom{a+s-1}{s-1} \zeta(s+t+a, r-a) + \sum_{a=0}^{s-1} \binom{a+r-1}{r-1} \zeta(r+t+a, s-a);$$

if $r+t > 0$ and $s+t > 0$, then

$$S(r, s, t) = \sum_{a=0}^{r-1} \binom{a+s-1}{s-1} \zeta(\overline{s+t+a}, r-a) + \sum_{a=0}^{s-1} \binom{a+r-1}{r-1} \zeta(\overline{r+t+a}, s-a);$$

if $r+t > 1$ and $s+t > 0$, then

$$R(r, s, t) = \sum_{a=0}^{r-1} \binom{a+s-1}{s-1} \zeta(\overline{s+t+a}, \overline{r-a}) + \sum_{a=0}^{s-1} \binom{a+r-1}{r-1} \zeta(r+t+a, \overline{s-a}).$$

Putting $t = 0$ in Theorem 1 yields the following decomposition formulas, the first of which was given under slightly more restrictive hypotheses in [15, Theorem 2.1].

Corollary 2. *If r and s are positive integers, then*

$$\begin{aligned}
\zeta_q[r] \zeta_q[s] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b \zeta_q[s+a, r-a-b] \\
&+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{r-1} \binom{r-1}{b} (1-q)^b \zeta_q[r+a, s-a-b] \\
&- \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \varphi[r+s-j];
\end{aligned}$$

$$\begin{aligned}
\zeta_q[\bar{r}]\zeta_q[\bar{s}] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b \zeta_q[\overline{s+a}, r-a-b] \\
&\quad + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{r-1} \binom{r-1}{b} (1-q)^b \zeta_q[\overline{r+a}, s-a-b] \\
&\quad - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \varphi[\overline{r+s-j}]; \\
\zeta_q[r]\zeta_q[\bar{s}] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b \zeta_q[\overline{s+a}, \overline{r-a-b}] \\
&\quad + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{r-1} \binom{r-1}{b} (1-q)^b \zeta_q[r+a, \overline{s-a-b}] \\
&\quad - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j (1+q)^{j-r-s} \zeta_{q^2}[r+s-j].
\end{aligned}$$

Taking the limit as $q \rightarrow 1+$ in Corollary 2 and noting the additional restrictions needed on r and s to guarantee convergence in this case yields the following decomposition formulas, the first of which was known to Euler.

Corollary 3. *If $r-1$ and $s-1$ are positive integers, then*

$$\zeta(r)\zeta(s) = \sum_{a=0}^{r-1} \binom{a+s-1}{s-1} \zeta(s+a, r-a) + \sum_{a=0}^{s-1} \binom{a+r-1}{r-1} \zeta(r+a, s-a);$$

if r and s are positive integers, then

$$\zeta(\bar{r})\zeta(\bar{s}) = \sum_{a=0}^{r-1} \binom{a+s-1}{s-1} \zeta(\overline{s+a}, r-a) + \sum_{a=0}^{s-1} \binom{a+r-1}{r-1} \zeta(\overline{r+a}, s-a);$$

if $r-1$ and s are positive integers, then

$$\zeta(r)\zeta(\bar{s}) = \sum_{a=0}^{r-1} \binom{a+s-1}{s-1} \zeta(\overline{s+a}, \overline{r-a}) + \sum_{a=0}^{s-1} \binom{a+r-1}{r-1} \zeta(r+a, \overline{s-a}).$$

3. PROOF OF THEOREM 1.

The key ingredient is the following partial fraction decomposition.

Lemma 1. *If $r, s, u,$ and v are all positive integers, then*

$$\begin{aligned} \frac{1}{[u]_q^r [v]_q^s} &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} \frac{(1-q)^b q^{(s-1-b)u+av}}{[u]_q^{r-a-b} [u+v]_q^{s+a}} \\ &+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} \frac{(1-q)^b q^{au+(r-1-b)v}}{[v]_q^{s-a-b} [u+v]_q^{r+a}} \\ &- \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} \frac{(1-q)^j q^{(s-j)u+(r-j)v}}{[u+v]_q^{r+s-j}}. \end{aligned}$$

Proof. Let x and y be nonzero real numbers such that $x + y + (q-1)xy \neq 0$. As in [15], observe that if we apply the partial differential operator

$$\frac{1}{(r-1)!} \left(-\frac{\partial}{\partial x} \right)^{r-1} \frac{1}{(s-1)!} \left(-\frac{\partial}{\partial y} \right)^{s-1}$$

to both sides of the identity

$$\frac{1}{xy} = \frac{1}{x+y+(q-1)xy} \left(\frac{1}{x} + \frac{1}{y} + q-1 \right),$$

then we obtain the identity [15, Lemma 3.1]

$$\begin{aligned} \frac{1}{x^r y^s} &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} \frac{(1-q)^b (1+(q-1)y)^a (1+(q-1)x)^{s-1-b}}{x^{r-a-b} (x+y+(q-1)xy)^{s+a}} \\ &+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} \frac{(1-q)^b (1+(q-1)x)^a (1+(q-1)y)^{r-1-b}}{y^{s-a-b} (x+y+(q-1)xy)^{r+a}} \\ &- \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} \frac{(1-q)^j (1+(q-1)y)^{r-j} (1+(q-1)x)^{s-j}}{(x+y+(q-1)xy)^{r+s-j}}. \end{aligned}$$

Now let $x = [u]_q$, $y = [v]_q$ and note that then $1 + (q-1)x = q^u$, $1 + (q-1)y = q^v$ and $x + y + (q-1)xy = [u+v]_q$. \square

To prove Theorem 1, multiply both sides of Lemma 1 by

$$\frac{\sigma^u \tau^v q^{(r+t-1)u+(s+t-1)v}}{[u+v]_q^t}$$

and sum over all ordered pairs of positive integers (u, v) to obtain

$$T[r, s, t; \sigma, \tau] = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \sum_{u,v=1}^{\infty} \frac{\sigma^u \tau^v q^{(r-a-b-1)u} q^{(s+t+a-1)(u+v)}}{[u]_q^{r-a-b} [u+v]_q^{s+t+a}}$$

$$\begin{aligned}
& + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \sum_{u,v=1}^{\infty} \frac{\sigma^u \tau^v q^{(s-a-b-1)v} q^{(r+t+a-1)(u+v)}}{[v]_q^{s-a-b} [u+v]_q^{r+t+a}} \\
& - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \sum_{u,v=1}^{\infty} \frac{\sigma^u \tau^v q^{(r+s+t-j-1)(u+v)}}{[u+v]_q^{r+s+t-j}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
T[r, s, t; \sigma, \sigma] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \sum_{m>n>0} \frac{\sigma^m q^{(s+t+a-1)m} q^{(r-a-b-1)n}}{[m]_q^{s+t+a} [n]_q^{r-a-b}} \\
& + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \sum_{m>n>0} \frac{\sigma^m q^{(r+t+a-1)m} q^{(s-a-b-1)n}}{[m]_q^{r+t+a} [n]_q^{s-a-b}} \\
& - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \sum_{m>n>0} \frac{\sigma^m q^{(r+s+t-j-1)m}}{[m]_q^{r+s+t-j}} \\
& = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \zeta_q[s+t+a, r-a-b; \sigma, 1] \\
& + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \zeta_q[r+t+a, s-a-b; \sigma, 1] \\
& - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \varphi[r+s+t-j; \sigma],
\end{aligned}$$

and also that

$$\begin{aligned}
T[r, s, t; 1, -1] &= \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \sum_{m>n>0} \frac{(-1)^m q^{(s+t+a-1)m} (-1)^n q^{(r-a-b-1)n}}{[m]_q^{s+t+a} [n]_q^{r-a-b}} \\
& + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \sum_{m>n>0} \frac{q^{(r+t+a-1)m} (-1)^n q^{(s-a-b-1)n}}{[m]_q^{r+t+a} [n]_q^{s-a-b}} \\
& - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \sum_{m>n>0} \frac{(-1)^n q^{(r+s+t-j-1)m}}{[m]_q^{r+s+t-j}} \\
& = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \zeta_q[s+t+a, r-a-b; \sigma, 1] \\
& + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b \zeta_q[r+t+a, s-a-b; \sigma, 1]
\end{aligned}$$

$$+ \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j \sum_{m>0} \frac{q^{(r+s+t-j-1)2m}}{[2m]_q^{r+s+t-j}}.$$

Since

$$[2m]_q = \frac{q^{2m} - 1}{q - 1} = \frac{q^2 - 1}{q - 1} \cdot \frac{q^{2m} - 1}{q^2 - 1} = (q + 1)[m]_{q^2},$$

the proof of Theorem 1 is complete. \square

4. EXAMPLES

Again, let $\sigma, \tau \in \{-1, 1\}$. It is known [4, 23] that if s and t are positive integers such that $s + t$ is odd and $s > (1 + \sigma)/2$, then $\zeta(s, t; \sigma, \tau)$ lies in the polynomial ring $\mathbf{Q}[\{\zeta(k; \pm 1) : k \in \mathbf{Z}, 2 \leq k \leq s + t\} \cup \{\zeta(1; -1)\}]$. Since

$$\zeta(k; -1) = \zeta(\bar{k}) = \begin{cases} (2^{1-k} - 1)\zeta(k) & \text{if } k > 1, \\ -\log 2 & \text{if } k = 1, \end{cases}$$

it is clear that in fact, $\zeta(s, t; \sigma, \tau) \in \mathbf{Q}[\{\zeta(k) : k \in \mathbf{Z}, 2 \leq k \leq s + t\} \cup \{-\log 2\}]$. It follows that if r, s, t satisfy the conditions of Corollary 1 and if in addition $r + s + t$ is an odd integer, then the signed and unsigned double Tornheim series $R(r, s, t)$, $S(r, s, t)$ and $T(r, s, t)$ also lie in this ring. It is possible to evaluate these series explicitly if we recall the following formula from [4, eq. (75)].

Proposition 1. *Let $\sigma, \tau \in \{-1, 1\}$, and let s and t be positive integers such that $s + t$ is odd, $s > (1 + \sigma)/2$, and $t > (1 + \tau)/2$. Then*

$$\begin{aligned} \zeta(s, t; \sigma, \tau) &= \frac{1}{2}(1 + (-1)^s)\zeta(s; \sigma)\zeta(t; \tau) - \frac{1}{2}\zeta(s + t; \sigma\tau) \\ &+ (-1)^t \sum_{0 \leq k \leq t/2} \binom{s+t-2k-1}{s-1} \zeta(2k; \sigma\tau)\zeta(s+t-2k; \sigma) \\ &+ (-1)^t \sum_{0 \leq k \leq s/2} \binom{s+t-2k-1}{t-1} \zeta(2k; \sigma\tau)\zeta(s+t-2k; \tau). \end{aligned} \quad (4)$$

In Proposition 1, it is understood that $\zeta(0; \sigma\tau) = -1/2$ in accordance with the analytic continuation of $s \mapsto \zeta(s; \sigma\tau)$. The restriction $t > (1 + \tau)/2$ can be removed if in (4) we interpret $\zeta(1; 1) = 0$ wherever it occurs. That is, if $\sigma \in \{-1, 1\}$ and s is an even positive integer, then

$$\zeta(s, 1; \sigma, 1) = \frac{1}{2}(s-1)\zeta(s+1; \sigma) + \frac{1}{2}\zeta(s+1) - \sum_{k=1}^{(s/2)-1} \zeta(2k; \sigma)\zeta(s+1-2k). \quad (5)$$

The case $\sigma = 1$ of (5) is subsumed by another formula [4, eq. (31)] of Euler, namely

$$\zeta(s, 1) = \frac{s}{2} \zeta(s+1) - \frac{1}{2} \sum_{k=2}^{s-1} \zeta(k) \zeta(s+1-k), \quad (6)$$

which is valid for *all* integers $s > 1$, not just for even s .

In [35], Tsumura listed evaluation formulas for $S(r, s, t)$ when $r+s+t \leq 9$ is odd. From Corollary 1, Proposition 1 and equation (5), we can deduce explicit formulas for $R(r, s, t)$, $S(r, s, t)$ and $T(r, s, t)$ when $r+s+t$ is odd. In particular, we have the following new results:

$$\begin{aligned} R(1, 1, 1) &= -\frac{5}{8}\zeta(3), & R(1, 1, 3) &= \frac{1}{16}\pi^2\zeta(3) - \frac{27}{32}\zeta(5), \\ R(1, 2, 2) &= \frac{5}{48}\pi^2\zeta(3) - \frac{3}{2}\zeta(5), & R(1, 3, 1) &= \frac{1}{12}\pi^2\zeta(3) - \frac{59}{32}\zeta(5), \\ R(2, 1, 2) &= -\frac{5}{16}\pi^2\zeta(3) + \frac{107}{32}\zeta(5), & R(2, 2, 1) &= -\frac{5}{24}\pi^2\zeta(3) + \frac{59}{32}\zeta(5), \\ R(3, 1, 1) &= \frac{1}{8}\pi^2\zeta(3) - \frac{59}{32}\zeta(5), \\ S(5, 5, 5) &= \frac{7}{73728}\pi^4\zeta(11) + \frac{35}{24576}\pi^2\zeta(13) + \frac{63}{8192}\zeta(15), \\ S(7, 7, 7) &= \frac{31}{35389440}\pi^6\zeta(15) + \frac{49}{1966080}\pi^4\zeta(17) + \frac{77}{262144}\pi^2\zeta(19) + \frac{429}{262144}\zeta(21). \end{aligned}$$

The values of $R(5, 5, 5)$, $R(7, 7, 7)$ and $R(9, 9, 9)$ listed in [34] appear to be incorrect. They should be

$$\begin{aligned} R(5, 5, 5) &= \frac{16375}{147456}\pi^4\zeta(11) + \frac{573335}{49152}\pi^2\zeta(13) - \frac{2064195}{16384}\zeta(15), \\ R(7, 7, 7) &= \frac{1048543}{70778880}\pi^6\zeta(15) + \frac{7339969}{3932160}\pi^4\zeta(17) + \frac{80740121}{524288}\pi^2\zeta(19) - \frac{899676921}{524288}\zeta(21), \\ R(9, 9, 9) &= \frac{13421747}{7046430720}\pi^8\zeta(19) + \frac{738197141}{2113929216}\pi^6\zeta(21) + \frac{1919313253}{67108864}\pi^4\zeta(23) \\ &\quad + \frac{143948506845}{67108864}\pi^2\zeta(25) - \frac{1631416447375}{67108864}\zeta(27). \end{aligned}$$

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