

ON THE SUM FORMULA FOR MULTIPLE q -ZETA VALUES

DAVID M. BRADLEY

ABSTRACT. Multiple q -zeta values are a one-parameter generalization (in fact, a q -analog) of the multiple harmonic sums commonly referred to as multiple zeta values. These latter are obtained from the multiple q -zeta values in the limit as $q \rightarrow 1$. Here, we discuss the sum formula for multiple q -zeta values, and provide a direct, self-contained proof. As a consequence, we also derive a q -analog of Euler's evaluation of the double zeta function $\zeta(m, 1)$.

1. Introduction. Sums of the form

$$(1) \quad \zeta(n_1, n_2, \dots, n_r) := \sum_{k_1 > k_2 > \dots > k_r > 0} \prod_{j=1}^r \frac{1}{k_j^{n_j}}$$

have attracted increasing attention in recent years, see e.g., [1–4, 6–8, 10, 11]. The survey articles [5, 19, 20] provide an extensive list of references. Here and throughout, n_1, \dots, n_r are positive integers with $n_1 > 1$, and we sum over all positive integers k_1, \dots, k_r satisfying the indicated inequalities. Note that, with positive integer arguments, $n_1 > 1$ is necessary and sufficient for convergence. The sums (1) are sometimes referred to as Euler sums, because they were first studied by Euler [12] in the case $r = 2$. In general, they may be profitably viewed as instances of the multiple polylogarithm [2, 5, 18], and are now more commonly referred to as multiple zeta values, reducing to the Riemann zeta function in the case $r = 1$. The present author introduced a q -analog of (1) in [9] as

$$(2) \quad \zeta[n_1, n_2, \dots, n_r] := \sum_{k_1 > k_2 > \dots > k_r > 0} \prod_{j=1}^r \frac{q^{(n_j-1)k_j}}{[k_j]_q^{n_j}},$$

AMS *Mathematics subject classification*. Primary 11M41, Secondary 11M06, 05A30, 33E20, 30B50.

Keywords and phrases. Multiple harmonic series, q -analog, multiple zeta values, q -series, Lambert series.

Received by the editors on February 28, 2005, and in revised form on March 8, 2005.

where

$$[k]_q := \sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}, \quad 0 < q < 1.$$

Observe that we now have

$$\zeta(n_1, \dots, n_r) = \lim_{q \rightarrow 1} \zeta_q[n_1, \dots, n_r],$$

so that (2) represents a generalization of (1). In this note, we prove an identity for (2), the $q = 1$ case of which was originally conjectured by Moen [14] and Market [16].

It is convenient to state results in terms of the shifted multiple q -zeta function defined by

$$\begin{aligned} \zeta^*[n_1, \dots, n_r] &:= \zeta[1 + n_1, n_2, \dots, n_r] \\ &= \sum_{k_1 > \dots > k_r > 0} \frac{q^{k_1}}{[k_1]_q} \prod_{j=1}^r \frac{q^{(n_j-1)k_j}}{[k_j]_q^{n_j}}. \end{aligned}$$

The main focus of our discussion is the following result.

Theorem 1. *If N and r are positive integers with $N \geq r$, then*

$$\sum_{\substack{n_1 + \dots + n_r = N \\ \forall j, n_j \geq 1}} \zeta^*[n_1, n_2, \dots, n_r] = \zeta^*[N],$$

where the sum is over all positive integers n_1, n_2, \dots, n_r such that $\sum_{j=1}^r n_j = N$.

The $q = 1$ case of Theorem 1 was proved for $r = 2$ by Euler, for $r = 3$ by Hoffman and Moen [15], and for general r by Granville [13]. The first proof of Theorem 1 in full generality, i.e., for general positive integer r and arbitrary $0 < q \leq 1$, was given by the present author [9], where it is derived as a consequence of other, deeper results for the multiple q -zeta function. Our purpose here is to give a simpler, more direct and self-contained proof of Theorem 1, and to exhibit a consequence of this result which generalizes another formula of Euler for the multiple zeta function.

2. Self-Contained proof of Theorem 1. By expanding both sides in powers of z and comparing coefficients, one readily sees that Theorem 1 is equivalent to the following result.

Theorem 2. *If r is a positive integer and $z \in \mathbf{C} \setminus \{q^{-m}[m]_q : m \in \mathbf{Z}^+\}$, then*

$$(3) \quad \sum_{k_1 > \dots > k_r > 0} \frac{q^{k_1}}{[k_1]_q} \prod_{j=1}^r \frac{1}{[k_j]_q - zq^{k_j}} = \sum_{m=1}^{\infty} \frac{q^{rm}}{[m]_q^r ([m]_q - zq^m)}.$$

Proof of Theorem 2. Let $L_r = L_r(z)$ denote the lefthand side of (3). By partial fractions,

$$(4) \quad L_r = \sum_{j=1}^r S_j$$

where

$$S_j = S_{j,r}(z) := \sum_{k_1 > \dots > k_r > 0} \frac{q^{k_1}}{[k_1]_q ([k_j]_q - zq^{k_j})} \prod_{\substack{i=1 \\ i \neq j}}^r \frac{1}{[k_i - k_j]_q}.$$

Now rename $k_j = m$ and sum first on m , so that

$$(5) \quad S_j = \sum_{m=1}^{\infty} \frac{A(m, j-1)B(m, r-j)}{[m]_q - zq^m},$$

where $A(m, 0) := q^m/[m]_q$,

$$A(m, j-1) := \sum_{k_1 > \dots > k_{j-1} > m} \frac{q^{k_1}}{[k_1]_q} \prod_{i=1}^{j-1} \frac{1}{[k_i - m]_q} \quad \text{for } 2 \leq j \leq r,$$

$B(m, 0) := 1$ and for $1 \leq j \leq r-1$,

$$\begin{aligned} B(m, r-j) &:= \sum_{m > k_{j+1} > \dots > k_r > 0} \prod_{i=j+1}^r \frac{1}{[k_i - m]_q} \\ &= (-1)^{r-j} \sum_{m > k_{j+1} > \dots > k_r > 0} \prod_{i=j+1}^r \frac{q^{m-k_i}}{[m - k_i]_q}. \end{aligned}$$

From (4) and (5) we now get that

$$L_r = \sum_{j=0}^{r-1} S_{j+1} = \sum_{m=1}^{\infty} \frac{1}{[m]_q - zq^m} \sum_{j=0}^{r-1} A(m, j)B(m, r - 1 - j),$$

and hence

$$(6) \quad \sum_{r=1}^{\infty} x^{r-1} L_r = \sum_{m=1}^{\infty} \frac{A_m(x)B_m(x)}{[m]_q - zq^m},$$

where the generating functions A_m and B_m are defined by

$$A_m(x) := \sum_{n=0}^{\infty} x^n A(m, n), \quad B_m(x) := \sum_{n=0}^{\infty} x^n B(m, n).$$

The proof of Theorem 2 now follows more or less immediately from the representations

$$(7) \quad A_m(x) = \frac{q^m}{[m]_q} \prod_{c=1}^m \left(1 - \frac{xq^c}{[c]_q}\right)^{-1} \quad \text{and} \quad B_m(x) = \prod_{b=1}^{m-1} \left(1 - \frac{xq^b}{[b]_q}\right).$$

To see this, observe that (7) gives

$$A_m(x)B_m(x) = \frac{q^m}{[m]_q} \left(1 - \frac{xq^m}{[m]_q}\right)^{-1} = \sum_{r=1}^{\infty} x^{r-1} \frac{q^r m}{[m]_q^r},$$

and hence from (6),

$$\sum_{r=1}^{\infty} x^{r-1} L_r = \sum_{r=1}^{\infty} x^{r-1} \sum_{m=1}^{\infty} \frac{q^r m}{[m]_q^r ([m]_q - zq^m)}.$$

It now remains only to prove the representations (7). First, note that

$$\begin{aligned} B_m(x) &= \sum_{n=0}^{\infty} x^n (-1)^n \sum_{m > k_1 > \dots > k_n > 0} \prod_{j=1}^n \frac{q^{m-k_j}}{[m-k_j]_q} \\ &= \sum_{n=0}^{\infty} (-x)^n \sum_{m > b_n > \dots > b_1 > 0} \prod_{j=1}^n \frac{q^{b_j}}{[b_j]_q} \\ &= \prod_{b=1}^{m-1} \left(1 - \frac{xq^b}{[b]_q}\right). \end{aligned}$$

Next, we define

$$A(m, n, k) := \sum_{b_1 > \dots > b_n > k} \frac{q^{m+b_1}}{[m+b_1]_q} \prod_{j=1}^n \frac{1}{[b_j]_q},$$

and note that $A(m, n) = A(m, n, 0)$. We have

$$\begin{aligned} A(m, 1, k) &= \sum_{b > k} \frac{q^{m+b}}{[m+b]_q [b]_q} \\ &= \frac{q^m}{[m]_q} \sum_{b > k} \left(\frac{q^b}{[b]_q} - \frac{q^{m+b}}{[m+b]_q} \right) \\ &= \frac{q^m}{[m]_q} \sum_{m \geq c \geq 1} \frac{q^{c+k}}{[c+k]_q}, \end{aligned}$$

and if, for some positive integer n ,

$$A(m, n, k) = \frac{q^m}{[m]_q} \sum_{m \geq c_1 \geq \dots \geq c_n \geq 1} \frac{q^{c_n+k}}{[c_n+k]_q} \prod_{j=1}^{n-1} \frac{q^{c_j}}{[c_j]_q},$$

then

$$\begin{aligned} A(m, n+1, k) &= \sum_{b_1 > \dots > b_{n+1} > k} \frac{q^{m+b_1}}{[m+b_1]_q} \prod_{j=1}^{n+1} \frac{1}{[b_j]_q} \\ &= \sum_{b_2 > \dots > b_{n+1} > k} \left(\prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) \sum_{b_1 > b_2} \frac{q^{m+b_1}}{[m+b_1]_q [b_1]_q} \\ &= \sum_{b_2 > \dots > b_{n+1} > k} \left(\prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) A(m, 1, b_2) \\ &= \sum_{b_2 > \dots > b_{n+1} > k} \left(\prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) \frac{q^m}{[m]_q} \sum_{c_0=1}^m \frac{q^{c_0+b_2}}{[c_0+b_2]_q} \\ &= \frac{q^m}{[m]_q} \sum_{c_0=1}^m \sum_{b_2 > \dots > b_{n+1} > k} \frac{q^{c_0+b_2}}{[c_0+b_2]_q} \prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \end{aligned}$$

$$\begin{aligned}
 &= \frac{q^m}{[m]_q} \sum_{c_0=1}^m A(c_0, n, k) \\
 &= \frac{q^m}{[m]_q} \sum_{m \geq c_0 \geq \dots \geq c_n \geq 1} \frac{q^{c_n+k}}{[c_n+k]_q} \prod_{j=0}^{n-1} \frac{q^{c_j}}{[c_j]_q},
 \end{aligned}$$

by the induction hypothesis. It follows that

$$A(m, n) = A(m, n, 0) = \frac{q^m}{[m]_q} \sum_{m \geq c_1 \geq \dots \geq c_n \geq 1} \prod_{j=1}^n \frac{q^{c_j}}{[c_j]_q},$$

and hence

$$\begin{aligned}
 A_m(x) &= \frac{q^m}{[m]_q} \prod_{c=1}^m \left(1 + \frac{xq^c}{[c]_q} + \left(\frac{xq^c}{[c]_q} \right)^2 + \left(\frac{xq^c}{[c]_q} \right)^3 + \dots \right) \\
 &= \frac{q^m}{[m]_q} \prod_{c=1}^m \left(1 - \frac{xq^c}{[c]_q} \right)^{-1}. \quad \square
 \end{aligned}$$

3. Evaluation of $\zeta[m, 1]$. Euler [12, 17] (see also [1, equation (31)]) proved that, for all integers $m \geq 2$,

$$2\zeta(m, 1) = m\zeta(m + 1) - \sum_{k=1}^{m-2} \zeta(m - k)\zeta(k + 1),$$

thereby expressing $\zeta(m, 1)$ in terms of values of the Riemann zeta function. The following q -analog of Euler’s formula, which the author proved in [9] using generating function techniques, is an easy consequence of the $r = 2$ case of Theorem 1 and the q -stuffle multiplication rule [9].

Corollary 1. *Let $2 \leq m \in \mathbf{Z}$. Then*

$$2\zeta[m, 1] = m\zeta[m + 1] + (1 - q)(m - 2)\zeta[m] - \sum_{k=1}^{m-2} \zeta[m - k]\zeta[k + 1].$$

In particular, when $m = 2$ we get $\zeta[2, 1] = \zeta[3]$, which is probably the simplest nontrivial identity satisfied by the multiple q -zeta function.

Proof of Corollary 1. For $1 \leq k \leq m - 2$ the q -stuffle multiplication rule [9] implies that

$$\zeta[m-k]\zeta[k+1] = \zeta[m+1] + (1-q)\zeta[m] + \zeta[m-k, k+1] + \zeta[k+1, m-k].$$

Summing on k , we find that

$$\sum_{k=1}^{m-2} \zeta[m-k]\zeta[k+1] = (m-2)(\zeta[m+1] + (1-q)\zeta[m]) + 2 \sum_{\substack{s+t=m+1 \\ s, t \geq 2}} \zeta[s, t].$$

But Theorem 1 gives

$$\sum_{\substack{s+t=m+1 \\ s, t \geq 2}} \zeta[s, t] = \sum_{\substack{s+t=m+1 \\ s \geq 2, t \geq 1}} \zeta[s, t] - \zeta[m, 1] = \zeta[m+1] - \zeta[m, 1].$$

It follows that

$$\sum_{k=1}^{m-2} \zeta[m-k]\zeta[k+1] = m\zeta[m+1] + (1-q)(m-2)\zeta[m] - 2\zeta[m, 1]. \quad \square$$

REFERENCES

1. J.M. Borwein, D.J. Broadhurst and D.M. Bradley, *Evaluations of k -fold Euler/Zagier sums: A compendium of results for arbitrary k* , Electron. J. Combin. **4** (1997), Wilf Festschrift.
2. J.M. Borwein, D.J. Broadhurst, D.M. Bradley and P. Lisoněk, *Special values of multiple polylogarithms*, Trans. Amer. Math. Soc. **353** (2001), 907–941. <http://arXiv.org/abs/math.CA/9910045>
3. ———, *Combinatorial aspects of multiple zeta values*, Electron. J. Combin. **5** (1998), #R38. <http://arXiv.org/abs/math.NT/9812020>
4. D. Bowman and D.M. Bradley, *Resolution of some open problems concerning multiple zeta evaluations of arbitrary depth*, Comp. Math. **139** (2003), 85–100. <http://arXiv.org/abs/math.CA/0310061>
5. ———, *Multiple polylogarithms: A brief survey*, B.C. Berndt and K. Ono, eds., Amer. Math. Soc. **291** (2001), 71–92. <http://arXiv.org/abs/math.CA/0310062>
6. ———, *The algebra and combinatorics of shuffles and multiple zeta values*, J. Combin. Theory **97** (2002), 43–61. <http://arXiv.org/abs/math.CO/0310082>

7. D. Bowman, D.M. Bradley and J. Ryoo, *Some multi-set inclusions associated with shuffle convolutions and multiple zeta values*, European J. Combin. **24** (2003), 121–127.
8. D.M. Bradley, *Partition identities for the multiple zeta function*, in *Zeta functions, topology, and physics*, Kinki University Mathematics Seminar Series, Developments in Mathematics, Vol. 14, T. Aoki, et al., eds., Springer-Verlag, 2005.
9. ———, *Multiple q -zeta values*, J. Algebra **283** (2005), 752–798. doi: 10.1016/j.jalgebra.2004.09.017 <http://arXiv.org/abs/math.QA/0402093>
10. ———, *Duality for finite multiple harmonic q -series*, Discrete Math. **300** (2005), 44–56.
11. David J. Broadhurst and Dirk Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B **393** (1997), 403–412.
12. L. Euler, *Meditationes Circa Singulare Serierum Genus*, Novi Comm. Acad. Sci. Petropol. **20** (1775), 140–186. Reprinted in *Opera Omnia*, Ser. I, **15**, B.G. Teubner, Berlin, 1927.
13. A. Granville, *A decomposition of Riemann's zeta-function*, in *Analytic number theory*, Cambridge University Press, Y. Motohashi, ed., 1997.
14. M.E. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), 275–290.
15. M.E. Hoffman and C. Moen, *Sums of triple harmonic series*, J. Number Theory **60** (1996), 329–331.
16. C. Markett, *Triple sums and the Riemann zeta function*, J. Number Theory **48** (1994), 113–132.
17. N. Nielsen, *Die Gammafunktion*, Chelsea, New York, 1965.
18. M. Waldschmidt, *Multiple polylogarithms: An introduction*, in *Number theory and discrete mathematics*, Hindustan Book Agency and Birkhäuser Verlag, 2002.
19. ———, *Valeurs zêta multiples: une introduction*, J. de Théorie des Nombres de Bordeaux **12** (2000), 581–595.
20. W. Zudilin [V.V. Zudilin], *Algebraic relations for multiple zeta values* Uspekhi Mat. Nauk **58** (2003), 3–32 (in Russian); Russian Math. Surveys **58** (2003), 1–29 (in English).

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF MAINE, 5752
 NEVILLE HALL, ORONO, MAINE 04469-5752
Email address: bradley@math.umaine.edu, dbradley@member.ams.org