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Multiple q-zeta values

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Abstract

We introduce a q-analog of the multiple harmonic series commonly referred to as multiple zeta values. The multiple q-zeta values satisfy a q-stuffle multiplication rule analogous to the stuffle multiplication rule arising from the series representation of ordinary multiple zeta values. Additionally, multiple q-zeta values can be viewed as special values of the multiple q-polylogarithm, which admits a multiple Jackson q-integral representation whose limiting case is the Drinfel'd simplex integral for the ordinary multiple polylogarithm when $q = 1$. The multiple Jackson q-integral representation for multiple q-zeta values leads to a second multiplication rule satisfied by them, referred to as a q-shuffle. Despite this, it appears that many numerical relations satisfied by ordinary multiple zeta values have no interesting q-extension. For example, a suitable q-analog of Broadhurst's formula for $\zeta(\{3,1\}^n)$, if one exists, is likely to be rather complicated. Nevertheless, we show that a number of infinite classes of relations, including Hoffman's partition identities, Ohno's cyclic sum identities, Granville's sum formula, Euler's convolution formula, Ohno's generalized duality relation, and the derivation relations of Ihara and Kaneko extend to multiple q-zeta values.

Keywords: Multiple harmonic series; q-Analog; Multiple zeta values; q-Series; Lambert series

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1. Introduction

Throughout, we assume q is real and $0 < q < 1$. The q-analog of a non-negative integer n is

$$[n]_q := \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}.$$

Definition 1. Let m be a positive integer and let $s_1, s_2, \dots, s_m$ be real numbers with $s_1 > 1$ and $s_j \geqslant 1$ for $2 \leqslant j \leqslant m$. The multiple q-zeta function is the nested infinite series

$$\zeta[s_1, \dots, s_m] := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}, \tag{1.1}$$

where the sum is over all positive integers k_j satisfying the indicated inequalities. If $m = 0$, the argument list in (1.1) is empty, and we define $\zeta[\,] := 1$. If the arguments in (1.1) are positive integers (with $s_1 > 1$ for convergence), we refer to (1.1) as a *multiple q-zeta value.*

Clearly, $\lim_{q\to 1} \zeta[s_1, \dots, s_m] = \zeta(s_1, \dots, s_m)$, where

$$\zeta(s_1, \dots, s_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}, \tag{1.2}$$

is the ordinary multiple zeta function [2–9,13,16]. In this paper, we make a detailed study of the multiple q-zeta function and its values at positive integer arguments. The q-stuffle rule and some of its implications are worked out in Section 2. Among other things, we derive a q-analog of the Newton recurrence [6, Eq. (4.5)] for $\zeta(\{s\}^n)$, a q-analog of Hoffman's partition identity [16, Theorem 2.2], [9], and a q-analog of the parity reduction theorem [3, Theorem 3.1]. In Section 3, we prove a q-analog of Ohno's generalized duality relation [25]. Consequences of our generalized q-duality relation include a q-analog of ordinary duality for multiple zeta values, and a q-analog of the sum formula [15]. In Section 4, we prove that the derivation theorem of Ihara and Kaneko [20] also extends to multiple q-zeta values. As we shall see, the q-analog of the Ihara–Kaneko derivation theorem is in fact equivalent to generalized q-duality. A special case ($n = 1$) yields a q-analog of Hoffman's derivation relation [16, Theorem 5.1], [19, Theorem 2.1]. In Section 5, we derive a q-analog of Ohno's cyclic sum formula [19]. In Section 6, we introduce the multiple q-polylogarithm, derive a Jackson q-integral analog of the Drinfel'd integral representation for ordinary multiple polylogarithms, and prove a q-analog of a formula [3, Theorem 9.1] for the colored multiple polylogarithm. Finally, in Section 7 we employ Heine's summation formula for the basic hypergeometric function to derive a bivariate generating function identity for the multiple q-zeta values $\zeta[m+2, \{1\}^n]$ $(0 \leqslant m,\ n \in \mathbf{Z})$. These are the values of the multiple q-zeta function evaluated at the indecomposable sequences [16] consisting of a positive integer greater than 1 followed by a string of n ones. Consequences of our generating function identity include the special case $\zeta[m+2,\{1\}^n] = \zeta[n+2,\{1\}^m]$ of q-duality, and a q-analog of Euler's evaluation expressing $\zeta(m+2,1)$ as a convolution of ordinary Riemann zeta values. More generally, we will see that for all integers $m \geqslant 2$ and $n \geqslant 0$, $\zeta[m,\{1\}^n]$ can be expressed in terms of q-zeta values of a single argument. Euler's formula is but a special case, as is Markett's formula [23] for $\zeta(m,1,1)$.

Whereas the structure of our arguments in many cases derives from the corresponding arguments in the classical $q=1$ case, the reader should not be surprised to learn that, as is often the case with those afflicted with a q-virus, much of the difficulty in establishing an appropriate q-theory is determining "where to put the q." In this light, it may be worth remarking that alternative definitions of the multiple q-zeta value are possible, and lead to other results. For example, in [10] we study the relationship between certain sums involving q-binomial coefficients with the finite sums

$$Z_n[s_1,\dots,s_m] := \sum_{n\geqslant k_1\geqslant\cdots\geqslant k_m\geqslant 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q^{s_j}}, \tag{1.3}$$

special cases of which have occurred in connection with some problems in sorting theory. Another model,

$$\zeta_q^*(s_1,\dots,s_m) := \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^m \frac{q^{k_j s_j}}{(1-q^{k_j})^{s_j}}$$

is suggested by Zudilin [30]. See also [28]. In Kaneko et al. [21], analytic properties of the q-analog

$$\zeta[s] = \sum_{k=1}^\infty \frac{q^{(s-1)k}}{[k]_q^s}$$

of the Riemann zeta function are studied. This immediately suggested Definition 1 to the present author. However, as we were subsequently informed, Zhao [29] had already been studying (1.1) and its polylogarithmic extension, albeit primarily from the viewpoint of analytic continuation and the q-shuffles of [6]. After a preliminary version (http://arXiv.org/abs/math.QA/0402093, v1, February 6, 2004) of the present paper was circulated, Okuda and Yoshihiro [27] re-outlined proofs of our Theorems 5 and 9 and gave a generalization of Theorem 15. For arithmetical results on single q-zeta values, see Zudilin [31–35].

Notation and terminology. As customary the boldface symbols $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{C}$ denote the sets of integers, rational numbers, and complex numbers, respectively. We will use $\mathbf{Z}^+$ for the set of positive integers; the subset $\{1,2,\dots,n\}$ consisting of the first n positive integers will be denoted by $\langle n\rangle$. We denote the cardinality of a set A by $|A|$, and when A is finite, the group of $|A|!$ permutations of $\langle |A|\rangle$ by $\mathfrak{S}(A)$. If $A=\langle n\rangle$, we write $\mathfrak{S}_n$ instead of $\mathfrak{S}(\langle n\rangle)$. Boolean expressions such as $(k\in A)$ take the value 1 if $k\in A$ and 0 if $k\notin A$. To avoid the potential for ambiguity in expressing complicated argument sequences without recourse to ellipses, we make occasional use of the abbreviations $\mathrm{Cat}_{j=1}^{m}\{s_j\}$ for the concatenated argument sequence $s_1, \dots, s_m$ and $\{s\}^m = \mathrm{Cat}_{j=1}^{m}\{s\}$ for $m \geqslant 0$ consecutive copies of s, which may itself be a sequence of arguments. Throughout, I will denote the set $\{0, 1\}$ and I^m the Cartesian product $I \times \cdots \times I$ of m copies of I when m is a positive integer. This will cause no confusion with the notation for concatenation, since we will never have occasion to discuss the periodic sequence $0, 1, \dots, 0, 1$. As in [3], we define the *depth* of the multiple q-zeta function (1.1) to be the number m of arguments.

2. q-Stuffles

The stuffle multiplication rule [3,6,9] for the multiple zeta function (also referred to as the harmonic product or $*$-product in [17,19]) arises when one expands the product of two nested series of the form (1.2), and is invariably given a recursive description. We begin with an explicit formula for the q-stuffle multiplication rule satisfied by the multiple q-zeta function; an explicit formula for the stuffle rule can then be derived by taking the limit as $q \to 1$.

Let m and n be positive integers. Define a *stuffle* on (m, n) as a pair (ϕ, ψ) of order-preserving injective mappings $\phi : \langle m\rangle \to \langle m + n\rangle$, $\psi : \langle n\rangle \to \langle m + n\rangle$ such that the union of their images is equal to $\langle r\rangle$ for some positive integer r with $\max(m, n) \leqslant r \leqslant m + n$. In what follows we will abuse notation by writing (for example) $\phi^{-1}(k)$ for the pre-image $\phi^{-1}(\{k\})$ of the singleton $\{k\}$. Since ϕ is injective, $\phi^{-1}(k)$ is either empty $\{\}$ or a singleton $\{j\}$ for some positive integer j, and we make the conventions

$$s_{\{j\}} = s_j, \qquad t_{\{j\}} = t_j, \qquad s_{\{\}} = t_{\{\}} = 0.$$

The stuffle multiplication rule for the multiple zeta function can now be written in the form

$$\zeta(s_1, \dots, s_m)\zeta(t_1, \dots, t_n) = \sum_{(\phi,\psi)} \zeta\left(\mathop{\mathrm{Cat}}_{k=1}^{r}\{s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)}\}\right), \tag{2.1}$$

where the sum is over all stuffles (ϕ, ψ) on (m, n), and $r = r(\phi, \psi)$ is the cardinality (equivalently, the largest member) of the union $\phi(\langle m\rangle) \cup \psi(\langle n\rangle)$ of the images of ϕ and ψ. More generally, expanding the product

$$\zeta[s_1, \dots, s_m]\zeta[t_1, \dots, t_n] = \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}} \sum_{l_1>\cdots>l_n>0} \prod_{j=1}^{n} \frac{q^{(t_j-1)l_j}}{[l_j]_q^{t_j}}$$

yields sums of products of terms of the form

$$\frac{q^{(s-1)k+(t-1)l}}{[k]_q^s [l]_q^t},$$

which, if $k = l$, reduces to

$$\frac{q^{(s+t-2)k}}{[k]_q^{s+t}} = (1-q)\frac{q^{(s+t-2)k}}{[k]_q^{s+t-1}} + \frac{q^{(s+t-1)k}}{[k]_q^{s+t}}.$$

It follows that

$$\begin{aligned}
&\zeta[s_1,\dots,s_m]\zeta[t_1,\dots,t_n] \\
&\quad = \sum_{(\phi,\psi)}\sum_{A}(1-q)^{|A|}\zeta\left[\operatorname*{Cat}_{k=1}^{r}\bigl\{s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)} - (k\in A)\bigr\}\right],
\end{aligned} \tag{2.2}$$

where the outer sum is over all stuffles (ϕ,ψ) on (m,n), the inner sum is over all subsets A of the intersection of the images of ϕ and ψ, $r = |\phi(\langle m\rangle)\cup\psi(\langle n\rangle)|$ as in (2.1), and the Boolean expression $(k\in A)$ takes the value 1 if $k\in A$ and 0 if $k\notin A$. We refer to (2.2) as the q-stuffle multiplication rule. Note that (2.1) is the limiting case $q\to 1$ of (2.2). For an alternative q-deformation of the stuffle algebra, see [18].

2.1. Period-1 sums completely reduce

As an application of the q-stuffle multiplication rule (2.2), we show that for any $s>1$ and positive integer n, the multiple q-zeta function $\zeta[\{s\}^n]$ can be expressed polynomially in terms of q-zeta functions of depth 1. See [5] for a discussion of the period-2 case for ordinary multiple zeta values and related alternating Euler sums.

Theorem 1. *If n is a positive integer and $s>1$, then*

$$n\zeta\bigl[\{s\}^n\bigr] = \sum_{k=1}^{n}(-1)^{k+1}\zeta\bigl[\{s\}^{n-k}\bigr]\sum_{j=0}^{k-1}\binom{k-1}{j}(1-q)^j\zeta[ks-j].$$

Proof. Let R denote the right-hand side of the equation in Theorem 1. The q-stuffle multiplication rule (2.2) implies that

$$\begin{aligned}
R = \sum_{k=1}^{n}(-1)^{k+1}\sum_{j=0}^{k-1}\binom{k-1}{j}(1-q)^j &\Biggl\{\sum_{m=0}^{n-k}\zeta\bigl[\{s\}^m, ks-j, \{s\}^{n-k-m}\bigr] \\
&+ \sum_{m=0}^{n-1-k}\zeta\bigl[\{s\}^m, (k+1)s-j, \{s\}^{n-1-k-m}\bigr] \\
&+ (1-q)\sum_{m=0}^{n-1-k}\zeta\bigl[\{s\}^m, (k+1)s-j-1, \{s\}^{n-1-k-m}\bigr]\Biggr\}.
\end{aligned} \tag{2.3}$$

Now expand (2.3) into three triple sums. We re-index the first and third of these, replacing k by $k+1$ in the first, and j by $j-1$ in the third. Then

$$\begin{aligned}
R = {} & \sum_{k=0}^{n-1}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(1-q)^j\sum_{m=0}^{n-1-k}\zeta\big[\{s\}^m,(k+1)s-j,\{s\}^{n-1-k-m}\big] \\
& + \sum_{k=1}^{n-1}(-1)^{k+1}\sum_{j=0}^{k-1}\binom{k-1}{j}(1-q)^j\sum_{m=0}^{n-1-k}\zeta\big[\{s\}^m,(k+1)s-j,\{s\}^{n-1-k-m}\big] \\
& + \sum_{k=1}^{n-1}(-1)^{k+1}\sum_{j=1}^{k}\binom{k-1}{j-1}(1-q)^j\sum_{m=0}^{n-1-k}\zeta\big[\{s\}^m,(k+1)s-j,\{s\}^{n-1-k-m}\big].
\end{aligned} \tag{2.4}$$

In the second and third triple sums (2.4), we have omitted the terms corresponding to $k = n$, because these vanish. In the second triple sum (2.4), the range on j can be extended to include the term $j = k$ because the binomial coefficient vanishes in that case. Similarly, the range on j in the third sum (2.4) can be extended to include the term $j = 0$. If we now combine the extended second and third triple sums (2.4) using the Pascal formula

$$\binom{k-1}{j}+\binom{k-1}{j-1}=\binom{k}{j},$$

we see that

$$\begin{aligned}
R = {} & \sum_{k=0}^{n-1}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(1-q)^j\sum_{m=0}^{n-1-k}\zeta\big[\{s\}^m,(k+1)s-j,\{s\}^{n-1-k-m}\big] \\
& + \sum_{k=1}^{n-1}(-1)^{k+1}\sum_{j=0}^{k}\binom{k}{j}(1-q)^j\sum_{m=0}^{n-1-k}\zeta\big[\{s\}^m,(k+1)s-j,\{s\}^{n-1-k-m}\big].
\end{aligned} \tag{2.5}$$

The two triple sums (2.5) cancel except for the $k = 0$ term in the first. Thus, we find that

$$R = \sum_{m=0}^{n-1}\zeta\big[\{s\}^m,s,\{s\}^{n-1-m}\big] = n\zeta\big[\{s\}^n\big],$$

as required. □

For reference, we note that letting $q \to 1$ in Theorem 1 yields the Newton recurrence [6, Eq. (4.5)] for multiple zeta values of period 1.

Corollary 1. *If n is a positive integer and $s > 1$, then*

$$n\zeta\big(\{s\}^n\big) = \sum_{k=1}^{n}(-1)^{k+1}\zeta\big(\{s\}^{n-k}\big)\zeta(ks).$$

2.2. Partition identities

Additional q-stuffle relations can be most easily stated using the concept of a set partition. As in [9], it is helpful to distinguish between set partitions that are ordered and those that are unordered.

Definition 2 (*Unordered set partition*). Let S be a finite non-empty set. An *unordered* set partition of S is a finite non-empty set $\mathcal{P}$ whose elements are disjoint non-empty subsets of S with union S. That is, there exists a positive integer $m = |\mathcal{P}|$ and non-empty subsets $P_1, \ldots, P_m$ of S such that $\mathcal{P} = \{P_1, \ldots, P_m\}$, $S = \bigcup_{k=1}^m P_k$, and $P_j \cap P_k$ is empty if $j \neq k$.

Definition 3 (*Ordered set partition*). Let S be a finite non-empty set. An *ordered* set partition of S is a finite ordered tuple $\vec{P}$ of disjoint non-empty subsets of S such that the union of the components of $\vec{P}$ is equal to S. That is, there exists a positive integer m and non-empty subsets $P_1, \ldots, P_m$ of S such that $\vec{P}$ can be identified with the ordered m-tuple $(P_1, \ldots, P_m)$, $\bigcup_{k=1}^m P_k = S$, and $P_j \cap P_k$ is empty if $j \neq k$.

We next introduce the shift operators E_k and δ_k defined as follows.

Definition 4. Let m and k be positive integers with $1 \leqslant k \leqslant m$, and let $s_1, \ldots, s_m$ be real numbers with $s_1 > 1$, $s_k \geqslant 2$, and $s_j \geqslant 1$ for $2 \leqslant j \neq k \leqslant m$. The shift operator E_k is defined by means of

$$E_k \zeta[s_1, \ldots, s_m] = \zeta\left[\operatorname*{Cat}_{j=1}^{k-1} s_j, s_k - 1, \operatorname*{Cat}_{j=k+1}^{m} s_j\right].$$

Let $\delta_k := \delta_k(q) = 1 + (1-q)E_k$ and abbreviate $\delta := \delta_1$.

The q-stuffle multiplication rule (2.2) can now be re-written in the form

$$\zeta[s_1, \ldots, s_m]\zeta[t_1, \ldots, t_n] = \sum_{(\phi,\psi)} \left(\prod_{k=1}^{r} \delta_k^{\alpha_k}\right) \zeta\left[\operatorname*{Cat}_{k=1}^{r} \{s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)}\}\right], \tag{2.6}$$

where $r = |\phi(\langle m\rangle) \cup \psi(\langle n\rangle)|$ and α_k is equal to 1 or 0 according as to whether k respectively is or is not a member of the intersection $\phi(\langle m\rangle) \cap \psi(\langle n\rangle)$ of the images of ϕ and ψ. Given (2.6), the following result is self-evident, but it can also be readily proved by mathematical induction.

Theorem 2. *Let n be a positive integer, and let $s_k > 1$ for $1 \leqslant k \leqslant n$. Then*

$$\prod_{k=1}^{n} \zeta[s_k] = \sum_{\vec{P} \vDash \langle n\rangle} \left(\prod_{j=1}^{|\vec{P}|} \delta_j^{|P_j|-1}\right) \zeta\left[\operatorname*{Cat}_{j=1}^{|\vec{P}|} \sum_{i \in P_j} s_i\right]$$

$$= \sum_{m=1}^{n} \sum_{\substack{\vec{P} \vDash \langle n \rangle \\ |\vec{P}|=m}} \sum_{\nu_1=0}^{|P_1|-1} \cdots \sum_{\nu_m=0}^{|P_m|-1} \zeta\left[\operatorname*{Cat}_{j=1}^{m} \left\{ \sum_{i \in P_j} s_i - \nu_j \right\} \right]$$

$$\times \prod_{j=1}^{m} \binom{|P_j|-1}{\nu_j} (1-q)^{\nu_j},$$

where the sum is over all ordered set partitions $\vec{P}$ of $\langle n \rangle$ having components $(P_1, \ldots, P_m)$, with $1 \leqslant m = |\vec{P}| \leqslant n$.

If in Theorem 2 we abbreviate $\sum_{i \in P_j} s_i$ by p_j and sum instead over unordered set partitions, we see that

$$\prod_{k=1}^{n} \zeta[s_k] = \sum_{\mathcal{P} \vdash \langle n \rangle} \left(\prod_{j=1}^{|\mathcal{P}|} \delta_j^{|P_j|-1} \right) \sum_{\sigma \in \mathfrak{S}(\mathcal{P})} \zeta\left[\operatorname*{Cat}_{j=1}^{|\mathcal{P}|} p_{\sigma(j)} \right], \tag{2.7}$$

where the $P_j \subseteq \langle n \rangle$ are the distinct disjoint members of $\mathcal{P}$. Inverting (2.7) and expanding the delta operators yields the following partition identity.

Theorem 3. *Let n be a positive integer, and let $s_j > 1$ for $1 \leqslant j \leqslant n$. Then*

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta\left[\operatorname*{Cat}_{j=1}^{n} s_{\sigma(j)} \right] = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|}$$

$$\times \prod_{k=1}^{|\mathcal{P}|} \big(|P_k|-1\big)! \sum_{\nu_k=0}^{|P_k|-1} \binom{|P_k|-1}{\nu_k} (1-q)^{\nu_k} \zeta[p_k - \nu_k],$$

where the sum on the right is over all unordered set partitions $\mathcal{P} = \{P_1, \ldots, P_m\}$ of $\langle n \rangle$, $1 \leqslant m = |\mathcal{P}| \leqslant n$, and $p_k = \sum_{j \in P_k} s_j$.

Letting $q \to 1$ in Theorem 3, we obtain the following result of Hoffman [16, Theorem 2.2], which he proved using a counting argument.

Corollary 2 (Hoffman's partition identity). *Let n be a positive integer, and let $s_j > 1$ for $1 \leqslant j \leqslant n$. Then*

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta\left(\operatorname*{Cat}_{j=1}^{n} s_{\sigma(j)} \right) = \sum_{\mathcal{P} \vdash \langle n \rangle} (-1)^{n-|\mathcal{P}|} \prod_{P \in \mathcal{P}} \big(|P|-1\big)!\, \zeta\left(\sum_{j \in P} s_j \right),$$

where the sum on the right is over all unordered set partitions $\mathcal{P}$ of $\langle n \rangle$.

Proof of Theorem 3. It is enough to show that

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\left[\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\right]=\sum_{\mathcal{P}\vdash\langle n\rangle}(-1)^{n-|\mathcal{P}|}\prod_{P\in\mathcal{P}}\bigl(|P|-1\bigr)!\,\delta^{|P|-1}\zeta\left[\sum_{j\in P}s_j\right]. \tag{2.8}$$

When $n=1$ this is trivial. Suppose the result (2.8) holds for $n-1$. Then

$$\sum_{\sigma\in\mathfrak{S}_{n-1}}\zeta\left[\operatorname*{Cat}_{j=1}^{n-1}s_{\sigma(j)}\right]=\sum_{\mathcal{P}\vdash\langle n-1\rangle}(-1)^{n-1-|\mathcal{P}|}\prod_{P\in\mathcal{P}}\bigl(|P|-1\bigr)!\,\delta^{|P|-1}\zeta\left[\sum_{j\in P}s_j\right]. \tag{2.9}$$

After multiplying Eq. (2.9) through by $\zeta[s_n]$, applying the q-stuffle multiplication rule (2.6) to the left-hand side, and moving the stuffed terms to the right, we obtain

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n}\zeta\left[\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\right]&=\sum_{\mathcal{P}\vdash\langle n-1\rangle}(-1)^{n-1-|\mathcal{P}|}\zeta[s_n]\prod_{P\in\mathcal{P}}\bigl(|P|-1\bigr)!\,\delta^{|P|-1}\zeta\left[\sum_{j\in P}s_j\right]\\&\quad-\sum_{\sigma\in\mathfrak{S}_{n-1}}\sum_{k=1}^{n-1}\delta_k\zeta\left[\operatorname*{Cat}_{j=1}^{k-1}s_{\sigma(j)},\,s_{\sigma(j)}+s_n,\,\operatorname*{Cat}_{j=k+1}^{n-1}s_{\sigma(j)}\right].\end{aligned} \tag{2.10}$$

Let $u_j^{(k)}=s_j$ if $j\neq k$ and $u_k^{(k)}=s_k+s_n$. With the aid of the inductive hypothesis (2.9), the double sum on the right-hand side of (2.10) can now be expressed in the form

$$\begin{aligned}\sum_{k=1}^{n-1}\sum_{\sigma\in\mathfrak{S}_n}\delta_{\sigma^{-1}(k)}\zeta\left[\operatorname*{Cat}_{j=1}^{n-1}u_{\sigma(j)}^{(k)}\right]&=\sum_{k=1}^{n-1}\sum_{\mathcal{P}\vdash\langle n-1\rangle}(-1)^{n-1-|\mathcal{P}|}\\&\quad\times\prod_{P\in\mathcal{P}}\bigl(|P|-1\bigr)!\,\delta^{|P|-1+(k\in P)}\zeta\left[\sum_{j\in P}u_j^{(k)}\right].\end{aligned}$$

From (2.10), it now follows that

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n}\zeta\left[\operatorname*{Cat}_{j=1}^{n}s_{\sigma(j)}\right]&=\sum_{\mathcal{P}\vdash\langle n-1\rangle}(-1)^{n-1-|\mathcal{P}|}\zeta[s_n]\prod_{P\in\mathcal{P}}\bigl(|P|-1\bigr)!\,\delta^{|P|-1}\zeta\left[\sum_{j\in P}s_j\right]\\&\quad+\sum_{k=1}^{n-1}\sum_{\mathcal{P}\vdash\langle n-1\rangle}(-1)^{n-|\mathcal{P}|}\prod_{P\in\mathcal{P}}\bigl(|P|-1\bigr)!\,\delta^{|P|-1+(k\in P)}\\&\quad\times\zeta\left[\sum_{j\in P}u_j^{(k)}\right].\end{aligned} \tag{2.11}$$

Note that in the second sum on the right-hand side of (2.11), as k runs from 1 to $n-1$, there is a contribution of $|P_0|$ copies of the inner sum if $P_0\in\mathcal{P}$ is such that $k\in P_0$. Therefore, if to each partition $\mathcal{P}$ of $\langle n-1\rangle$ in the first sum on the right-hand side of (2.11), we let $\mathcal{R}=\mathcal{P}\cup\{\{n\}\}$, then

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\left[\operatorname*{Cat}_{j=1}^{n} s_{\sigma(j)}\right]=\sum_{\substack{\mathcal{R}\vdash\langle n\rangle\\ \{n\}\in\mathcal{R}}}(-1)^{n-|\mathcal{R}|}\prod_{R\in\mathcal{R}}\big(|R|-1\big)!\,\delta^{|R|-1}\zeta\left[\sum_{j\in R}s_j\right]$$
$$+\sum_{\substack{\mathcal{P}\vdash\langle n-1\rangle\\ P_0\in\mathcal{P}}}(-1)^{n-|\mathcal{P}|}|P_0|!\,\delta^{|P_0|}\zeta\left[s_n+\sum_{j\in P_0}s_j\right]$$
$$\times\prod_{\substack{P\in\mathcal{P}\\ P\neq P_0}}\big(|P|-1\big)!\,\delta^{|P|-1}\zeta\left[\sum_{j\in P}s_j\right]. \tag{2.12}$$

Clearly, the second sum on the right-hand side of (2.12) can be re-written more succinctly if we simply toss n into P_0 and thus view each $\mathcal{P}$ as an unordered set partition of $\langle n\rangle$ in which no part in the partition is equal to the singleton $\{n\}$. Thus,

$$\sum_{\sigma\in\mathfrak{S}_n}\zeta\left[\operatorname*{Cat}_{j=1}^{n} s_{\sigma(j)}\right]=\sum_{\substack{\mathcal{R}\vdash\langle n\rangle\\ \{n\}\in\mathcal{R}}}(-1)^{n-|\mathcal{R}|}\prod_{R\in\mathcal{R}}\big(|R|-1\big)!\,\delta^{|R|-1}\zeta\left[\sum_{j\in R}s_j\right]$$
$$+\sum_{\substack{\mathcal{P}\vdash\langle n\rangle\\ \{n\}\notin\mathcal{P}}}(-1)^{n-|\mathcal{P}|}\prod_{P\in\mathcal{P}}\big(|P|-1\big)!\,\delta^{|P|-1}\zeta\left[\sum_{j\in P}s_j\right].$$

The result (2.8) now follows, since any partition of $\langle n\rangle$ is either of the form $\mathcal{R}$ or $\mathcal{P}$ above. □

Remark 1. The proof shows that Theorem 3 (and hence also its limiting case, Corollary 2) relies on only the q-stuffle multiplication property. Loosely speaking, we refer to results such as Theorems 2 and 3 and Corollary 2 as *partition identities* because they are easily stated using the language of set partitions. The notion is defined precisely in [9], where among other things it is shown that *all* partition identities are a consequence of the stuffle multiplication rule, and hence a decision procedure exists for verifying them.

We conclude this section with one further result, namely a q-analog of [3, Theorem 3.1]. Results which go beyond stuffles will be discussed in the subsequent sections.

Theorem 4 (Parity reduction). *Let m be a positive integer, and let $s_1,\dots,s_m$ be real numbers with $s_1>1$, $s_m>1$, and $s_j\geqslant 1$ for $1<j<m$. Then*

$$\zeta\left[\operatorname*{Cat}_{k=1}^{m} s_k\right]+(-1)^m\zeta\left[\operatorname*{Cat}_{k=1}^{m} s_{m-k+1}\right]$$

can be expressed as a $\mathbf{Z}[q]$*-linear combination of multiple* q*-zeta values of depth less than* m*. That is, the coefficients in the linear combination are polynomials in* q *with integer coefficients.*

Proof. Let N denote the Cartesian product of m copies of the positive integers. Define an additive weight-function on subsets of N by

$$w(A) := \sum_{\vec{n}\in A} \prod_{k=1}^{m} \frac{q^{(s_k-1)n_k}}{[n_k]_q^{s_k}},$$

where the sum is over all $\vec{n} = (n_1, \dots, n_m) \in A$. For each $k \in \langle m-1\rangle$, define the subset P_k of N by $P_k = \{\vec{n} \in N\colon\ n_k \leqslant n_{k+1}\}$. The Inclusion–Exclusion Principle states that

$$w\Biggl(\bigcap_{k=1}^{m-1} N \setminus P_k\Biggr) = \sum_{T\subseteq\langle m-1\rangle} (-1)^{|T|} w\Biggl(\bigcap_{k\in T} P_k\Biggr). \tag{2.13}$$

The term on the right-hand side of (2.13) arising from the empty subset $T = \{\}$ is $\prod_{k=1}^{m} \zeta[s_k]$ by the usual convention for intersection over an empty set. The left-hand side of (2.13) is simply $\zeta[s_1, \dots, s_m]$. In light of the identity

$$\frac{q^{(s-2)n}}{[n]_q^s} = \frac{q^{(s-1)n}}{[n]_q^s} + (1-q)\frac{q^{(s-2)n}}{[n]_q^{s-1}},$$

it follows that the right-hand side of (2.13) is a $\mathbf{Z}[q]$-linear combination of multiple q-zeta values of depth strictly less than m, except for the term corresponding to $T = \langle m-1\rangle$, which contributes

$$\begin{aligned}
&(-1)^{m-1} \sum_{1\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_m} \prod_{k=1}^{m} \frac{q^{(s_k-1)n_k}}{[n_k]_q^{s_k}} \\
&\quad = (-1)^{m-1}\zeta\Bigl[\operatorname*{Cat}_{k=1}^{m} s_{m-k+1}\Bigr] \\
&\qquad + \bigl(\mathbf{Z}[q]\text{-linear combination of lower depth multiple } q\text{-zeta values}\bigr). \qquad \square
\end{aligned}$$

3. Generalized q-duality

In this section, we prove a q-analog of Ohno's generalized duality relation [25]. As a consequence, we derive q-analogs of the duality relation [2,3,6,16,17] and the sum formula [15]. An additional consequence is a q-analog of Ihara and Kaneko's derivation theorem [20], which we prove in Section 4.

Definition 5. Let n and $s_1, \dots, s_n$ be positive integers with $s_1 > 1$. Let m be a non-negative integer. Define

$$Z[s_1, \dots, s_n; m] := \sum_{\substack{c_1,\dots,c_n \geqslant 0 \\ c_1+\cdots+c_n=m}} \zeta[s_1 + c_1, \dots, s_n + c_n],$$

where the sum is over all non-negative integers c_j with $\sum_{j=1}^n c_j = m$. As in [2], for non-negative integers a_j and b_j, define the dual argument lists

$$\mathbf{p} = \left(\operatorname*{Cat}_{j=1}^{n} \left\{ a_j + 2, \{1\}^{b_j} \right\} \right), \qquad \mathbf{p}' = \left(\operatorname*{Cat}_{j=1}^{n} \left\{ b_{n-j+1} + 2, \{1\}^{a_{n-j+1}} \right\} \right).$$

Theorem 5 (Generalized q-duality). *For any pair of dual argument lists* $\mathbf{p}$, $\mathbf{p}'$ *and any non-negative integer* m, *we have the equality* $Z[\mathbf{p}; m] = Z[\mathbf{p}'; m]$.

The $m = 0$ case of Theorem 5 is worth stating separately. It is a direct q-analog of the duality relation for multiple zeta values. A related, but distinct duality result for (1.3) is proved in [10].

Corollary 3 (q-Duality). *For any pair of dual argument lists* $\mathbf{p}$ *and* $\mathbf{p}'$, *we have the equality* $\zeta[\mathbf{p}] = \zeta[\mathbf{p}']$. *In other words, for all non-negative integers* a_j, b_j, $1 \leqslant j \leqslant n$, *we have the equality*

$$\zeta\left[\operatorname*{Cat}_{j=1}^{n} \left\{ a_j + 2, \{1\}^{b_j} \right\} \right] = \zeta\left[\operatorname*{Cat}_{j=1}^{n} \left\{ b_{n-j+1} + 2, \{1\}^{a_{n-j+1}} \right\} \right].$$

As noted by Ohno [25], the sum formula [15] is an easy consequence of his generalized duality relation. Likewise, the following q-analog of the sum formula is a consequence of our generalized q-duality relation (Theorem 5).

Corollary 4 (q-Sum formula). *For any integers* $0 < k \leqslant n$, *we have*

$$\sum_{s_1+s_2+\cdots+s_n=k} \zeta[s_1 + 1, s_2, \dots, s_n] = \zeta[k + 1],$$

where the sum is over all positive integers $s_1, s_2, \dots, s_n$ *with sum equal to* k.

Proof. If we take the dual argument lists in the form $\mathbf{p} = (n + 1)$ and $\mathbf{p}' = (2, \{1\}^{n-1})$ and put $m = k - n$, then Theorem 5 states that

$$\zeta[k + 1] = \sum_{\substack{c_1,\dots,c_n \geqslant 0 \\ c_1+\cdots+c_n=k-n}} \zeta\left[2 + c_2, \operatorname*{Cat}_{j=2}^{n} \{1 + c_j\} \right] = \sum_{\substack{s_1,\dots,s_n \geqslant 1 \\ s_1+\cdots+s_n=k}} \zeta\left[s_1 + 1, \operatorname*{Cat}_{j=2}^{n} s_j \right]. \qquad \square$$

Remark 2. The q-sum formula (Corollary 4) is also easily seen to be equivalent to the identity

$$\sum_{k_1>\cdots>k_n>0} \frac{q^{k_1}}{[k_1]_q} \prod_{j=1}^{n} \frac{1}{[k_j]_q - zq^{k_j}} = \sum_{m=1}^{\infty} \frac{q^{nm}}{[m]_q^n([m]_q - zq^m)}, \quad n \in \mathbf{Z}^+,\ z \in \mathbf{C},$$

which is given an independent proof in [11].

3.1. Proof of generalized q-duality

To prove Theorem 5, we need to employ some algebraic machinery first introduced by Hoffman [17]. The argument itself extends ideas of Okuda and Ueno [26] to the q-case. Let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y, and let $\mathfrak{h}^0$ denote the subalgebra $\mathbf{Q}1 \oplus x\mathfrak{h}y$. The $\mathbf{Q}$-linear map $\hat{\zeta} : \mathfrak{h}^0 \to \mathbf{R}$ is defined by $\hat{\zeta}[1] := \zeta[\,] = 1$ and

$$\hat{\zeta}\left[\prod_{j=1}^{s} x^{a_j} y^{b_j}\right] = \zeta\left[\operatorname*{Cat}_{j=1}^{s}\bigl\{a_j + 1, \{1\}^{b_j-1}\bigr\}\right], \quad a_j, b_j \in \mathbf{Z}^+.$$

For each positive integer n, let D_n be the derivation on $\mathfrak{h}$ that maps $x \mapsto 0$ and $y \mapsto x^n y$, and let θ be a formal parameter. Then $\sum_{n=1}^{\infty} D_n\theta^n/n$ is a derivation on $\mathfrak{h}[\![\theta]\!]$ and $\sigma_\theta = \exp(\sum_{n=1}^{\infty} D_n\theta^n/n)$ is an automorphism of $\mathfrak{h}[\![\theta]\!]$. Let τ be the anti-automorphism of $\mathfrak{h}$ that switches x and y. For any word $w \in \mathfrak{h}^0$, define $f[w;\theta] := \hat{\zeta}[\sigma_\theta(w)]$ and $g[w;\theta] := \hat{\zeta}[\sigma_\theta(\tau(w))]$. By definition of D_n, $\sum_{n=1}^{\infty} D_n\theta^n/n$ sends $x \mapsto 0$ and $y \mapsto \{\log(1-x\theta)^{-1}\}y$. Thus, σ_θ sends $x \mapsto x$ and $y \mapsto (1-x\theta)^{-1}y$. Therefore,

$$\begin{aligned} f\left[\prod_{j=1}^{s} x^{a_j} y^{b_j};\theta\right] &= \hat{\zeta}\left[\prod_{j=1}^{s} x^{a_j}\bigl\{(1-x\theta)^{-1}y\bigr\}^{b_j}\right] \\ &= \sum_{m=0}^{\infty} \theta^m \sum_{\substack{c_1,\dots,c_n \geqslant 0\\ c_1+\cdots+c_n=m}} \zeta\left[\operatorname*{Cat}_{i=1}^{n}\{k_i + c_i\}\right], \end{aligned} \tag{3.1}$$

where $(k_1,\dots,k_n) = (\operatorname{Cat}_{j=1}^{s}\{a_j+1, \{1\}^{b_j-1}\})$ and $n = \sum_{j=1}^{s} b_j$. Theorem 5 can now be restated in the equivalent form given below.

Theorem 6 (Generalized q-duality, reformulated). *For all $w \in \mathfrak{h}^0$, $f[w;\theta] = g[w;\theta]$. In other words, $\hat{\zeta} \circ \sigma_\theta$ is invariant under ordinary duality τ.*

The following difference equation is a key result in the proof of Theorem 6.

Theorem 7. *Let a_i, b_i be positive integers with $\sum_{i=1}^{s}(a_i + b_i) > 2$. Make the abbreviation $\theta' := q\theta - 1$, and recall the notation $I^m = \{0,1\} \times \cdots \times \{0,1\}$ for the m-fold Cartesian product from Section* 1. *The generating functions f and g satisfy the difference equation*

$$\sum_{\substack{\epsilon,\delta\in I^s\\ \delta_1<a_1,\ \epsilon_s<b_s}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i};\theta\right]$$
$$= \sum_{\substack{\delta,\epsilon\in I^{s+1}\\ \delta_{s+1}=\epsilon_1=0\\ \delta_1<a_1,\ \epsilon_{s+1}<b_s}} (-\theta')^{\bar{\delta}\cdot\bar{\epsilon}-1}(1-q)^{\delta\cdot\epsilon} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_{i+1}};\theta'\right].$$

Here, we use $\bar{\delta}$ to denote the ordered tuple whose ith component is $1-\delta_i$, and of course $\delta\cdot\epsilon$ denotes the dot product $\sum_i \delta_i\epsilon_i$. Similarly, $\bar{\epsilon}$ denotes the ordered tuple whose ith component is $1-\epsilon_i$, and $\bar{\delta}\cdot\bar{\epsilon}=\sum_i(1-\delta_i)(1-\epsilon_i)$.

We also require the following lemma, which shows that the generating function $f[w;\theta]$ can be analytically continued to a meromorphic function of θ with at worst simple poles at $\theta = q^{-\nu}[\nu]_q$ for positive integers ν.

Lemma 1. *Let $w=\prod_{i=1}^{s} x^{a_i} y^{b_i}$, where a_i and b_i are positive integers. Let $B_0:=0$ and set $B_i:=\sum_{j=1}^{i} b_j$ for $1\leqslant i\leqslant s$. Then*

$$f[w;\theta]=\sum_{\nu=1}^{\infty}\frac{C_\nu[w]}{[\nu]_q-\theta q^{\nu}},$$

where

$$C_\nu[w]:=\sum_{k=1}^{B_s}\sum_{\substack{m_1>\cdots>m_{k-1}>\nu\\ \nu>m_{k+1}>\cdots>m_{B_s}>0}} E_k[w;m_1,\dots,m_{k-1},\nu,m_{k+1},\dots,m_{B_s}],$$

and

$$E_k[w;m_1,\dots,m_{k-1},\nu,m_{k+1},\dots,m_{B_s}]$$
$$=\left\{\prod_{i=1}^{s}\frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}\right\}\Bigg/\prod_{\substack{j=1\\ j\neq k}}^{B_s}\big([m_j]_q-q^{m_j-\nu}[\nu]_q\big).$$

In the expression for E_k, we have placed the compound subscript $1+B_{i-1}$ in parentheses to emphasize that the entire expression $1+B_{i-1}$ occurs in the subscript of m.

We defer the proofs of Theorem 7 and Lemma 1 in order to proceed directly to the proof of Theorem 6.

Proof of Theorem 6. We use induction on the total degree of the word $\prod_{i=1}^{s} x^{a_i} y^{b_i}$. The base case is clearly satisfied, since the word xy is self-dual. Now apply Theorem 7 to f and g. Subtracting the two equations gives

$$\sum_{\substack{\delta,\epsilon\in I^s\\ \delta_1<a_1,\ \epsilon_s<b_s}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon}\left\{f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i};\theta\right]-g\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i};\theta\right]\right\}$$

$$= \sum_{\substack{\delta,\epsilon\in I^{s+1}\\ \delta_{s+1}=\epsilon_1=0\\ \delta_1<a_1,\ \epsilon_{s+1}<b_s}} (-\theta')^{\bar{\delta}\cdot\bar{\epsilon}-1}(1-q)^{\delta\cdot\epsilon}$$

$$\times\left\{f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_{i+1}};\theta'\right]-g\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_{i+1}};\theta'\right]\right\}.$$

But the terms whose words have total degree less than $\sum_{i=1}^{s}(a_i+b_i)$ are cancelled by the induction hypothesis. This leaves us with

$$(-\theta)^s\left\{f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta\right]-g\left[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta\right]\right\}$$

$$=(-\theta')^s\left\{f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta'\right]-g\left[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta'\right]\right\}.$$

Thus, the function

$$H(\theta):=(-\theta)^s\left\{f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta\right]-g\left[\prod_{i=1}^{s} x^{a_i} y^{b_i};\theta\right]\right\}$$

satisfies the functional equation $H(\theta)=H(\theta')$, where $\theta'=q\theta-1$. But by Lemma 1, $H(\theta)$ is a meromorphic function of θ of the form

$$\theta^s\sum_{\nu=1}^{\infty}\frac{h_\nu}{[\nu]_q-\theta q^\nu},$$

with at worst simple poles at $\theta=p_\nu:=q^{-\nu}[\nu]_q$ for positive integers ν. Note that $0=p_0<p_1<p_2<\cdots$ and $p'_\nu=qp_\nu-1=p_{\nu-1}$ for all $\nu\geqslant 1$. The functional equation thus implies that if H has a pole at p_ν, then H must also have a pole at $p_{\nu-1}$. Since H has no pole at p_0, it follows that each $h_\nu=0$. Thus, H vanishes identically and the proof is complete. □

Let $1\neq w=\prod_{i=1}^{s} x^{a_i} y^{b_i}\in\mathfrak{h}^0$. Henceforth, we assume that $|\theta|<1/q$. To prove that f and g satisfy the difference equation as stipulated by Theorem 7, first observe that from (3.1),

$$
\begin{aligned}
f[w;\theta] &= \sum_{\nu=0}^{\infty}\theta^{\nu} \sum_{\substack{c_j\geqslant 0\\ \sum_{j=1}^{n} c_j=\nu}} \sum_{m_1>\cdots>m_n>0} \prod_{j=1}^{n} \frac{q^{(k_j+c_j-1)m_j}}{[m_j]_q^{k_j+c_j}} \\
&= \sum_{m_1>\cdots>m_n>0} \prod_{j=1}^{n} \sum_{c_j=0}^{\infty} \frac{q^{(k_j+c_j-1)m_j}}{[m_j]_q^{k_j+c_j}}\theta^{c_j} \\
&= \sum_{m_1>\cdots>m_n>0} \prod_{j=1}^{n} \frac{q^{(k_j-1)m_j}}{[m_j]_q^{k_j-1}([m_j]_q-\theta q^{m_j})} \\
&= \sum_{m_1>\cdots>m_{B_s}>0} \prod_{i=1}^{s} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q-\theta q^{m_j}},
\end{aligned}
\tag{3.2}
$$

where $B_0 := 0$ and $B_i := \sum_{j=1}^{i} b_j$ for $1 \leqslant i \leqslant s$ as in the statement of Lemma 1.

Definition 6. If $d = (d_1, \ldots, d_s) \in I^s$ is such that $d_s = 0$ if $b_s = 1$, let

$$
f[w;d;\theta] := \sum_{m_1>\cdots>m_{B_s}>0} \prod_{i=1}^{s} \frac{q^{a_i(m_{(1+B_{i-1})}-d_i)}}{[m_{(1+B_{i-1})}-d_i]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q-\theta q^{m_j}}.
$$

The extra requirement on d_s ensures that no division by zero occurs when $B_s = 1$. Note that we now have $f[w;\theta] = f[w;\{0\}^s;\theta]$. For the proof of Theorem 7, we require the following sequence of lemmata.

Lemma 2. *If* $(s > 1$ *or* $b_1 > 1)$ *and* $a_1 > 1$*, then*

$$
\begin{aligned}
&\sum_{\delta,\epsilon\in I} (-\theta)^{\bar\delta\bar\epsilon}(1-q)^{\delta\epsilon} f\Bigg[x^{a_1-\delta}y^{b_1-\epsilon}\prod_{i=2}^{s} x^{a_i}y^{b_i};\{0\}^s;\theta\Bigg] \\
&\quad = \sum_{\delta\in I} (-\theta')^{\bar\delta} f\Bigg[x^{a_1-\delta}y^{b_1}\prod_{i=2}^{s} x^{a_i}y^{b_i};1,\{0\}^{s-1};\theta\Bigg].
\end{aligned}
$$

Lemma 3. *If* $s > 1$ *or* $b_1 > 1$*, then*

$$
\sum_{\epsilon\in I} (-\theta)^{\bar\epsilon} f\Bigg[xy^{b_1-\epsilon}\prod_{i=2}^{s} x^{a_i}y^{b_i};\{0\}^s;\theta\Bigg] = (-\theta') f\Bigg[xy^{b_1}\prod_{i=2}^{s} x^{a_i}y^{b_i};1,\{0\}^{s-1};\theta\Bigg].
$$

Lemma 4. *If* $1 < j < s$ *or* $(j = s$ *and* $b_s > 1)$*, then*

$$
\sum_{\delta,\epsilon\in I} (-\theta)^{\bar\delta\bar\epsilon}(1-q)^{\delta\epsilon} f\Bigg[\Bigg(\prod_{i=1}^{j-1} x^{a_i}y^{b_i}\Bigg)x^{a_j-\delta}y^{b_j-\epsilon}\prod_{i=j+1}^{s} x^{a_i}y^{b_i};\{1\}^{j-1},\{0\}^{s-j+1};\theta\Bigg]
$$

$$= \sum_{\delta,\epsilon\in I} (-\theta')^{\bar{\delta}\bar{\epsilon}}(1-q)^{\delta\epsilon}$$
$$\times f\left[\left(\prod_{i=1}^{j-2} x^{a_i}y^{b_i}\right) x^{a_{j-1}}y^{b_{j-1}-\epsilon}x^{a_j-\delta}y^{b_j}\prod_{i=j+1}^{s} x^{a_i}y^{b_i}; \{1\}^j, \{0\}^{s-j}; \theta\right].$$

Lemma 5. *If* $b_s > 1$, *then*

$$f\left[\prod_{i=1}^{s} x^{a_i}y^{b_i}; \{1\}^s; \theta\right] = \sum_{\epsilon\in I}(-\theta')^{-\epsilon} f\left[\left(\prod_{i=1}^{s-1} x^{a_i}y^{b_i}\right) x^{a_s}y^{b_s-\epsilon}; \{0\}^s; \theta'\right].$$

Lemma 6. *If* $s > 1$, *then*

$$\sum_{\delta\in I}(-\theta)^{\bar{\delta}} f\left[\left(\prod_{i=1}^{s-1} x^{a_i}y^{b_i}\right) x^{a_s-\delta}y; \{1\}^{s-1}; 0; \theta\right]$$
$$= \sum_{\delta,\epsilon\in I} (-\theta')^{\bar{\delta}\bar{\epsilon}}(1-q)^{\delta\epsilon} f\left[\left(\prod_{i=1}^{s-2} x^{a_i}y^{b_i}\right) x^{a_{s-1}}y^{b_{s-1}-\epsilon}x^{a_s-\delta}y; \{0\}^s; \theta'\right].$$

Lemma 7. *If* $a > 1$, *then*

$$\sum_{\delta\in I}(-\theta)^{\bar{\delta}} f\big[x^{a-\delta}y; \theta\big] = \sum_{\delta\in I}(-\theta')^{\bar{\delta}} f\big[x^{a-\delta}y; \theta'\big].$$

For completeness, we also record the following result, although it is not needed for the proof of Theorem 7.

Lemma 8. $\theta f[xy;\theta] + (1-q) = \theta' f[xy;\theta'] - 1/\theta'$.

We shall prove Lemmas 1–8 in Section 3.2 below. Assuming their validity for now, we proceed with the proof of Theorem 7.

Proof of Theorem 7. Let L denote the left-hand side. First, consider the case when $a_1 > 1$ and $b_s > 1$. Then

$$L = \sum_{\delta,\epsilon\in I^s} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i}y^{b_i-\epsilon_i}; \theta\right].$$

In the sum over ordered s-tuples δ and ϵ, rename $\delta = (\delta_2, \dots, \delta_s)$ and $\epsilon = (\epsilon_2, \dots, \epsilon_s)$ so that

$$
\begin{aligned}
L &= \sum_{\delta,\epsilon\in I^{s-1}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} \sum_{\delta_1,\epsilon_1\in I} (-\theta)^{\bar{\delta}_1\bar{\epsilon}_1}(1-q)^{\delta_1\epsilon_1} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i}; \{0\}^s; \theta\right] \\
&= \sum_{\delta,\epsilon\in I^{s-1}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} \sum_{\delta_1\in I} (-\theta')^{\bar{\delta}_1} f\left[x^{a_1-\delta_1} y^{b_1} \prod_{i=2}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i}; 1, \{0\}^{s-1}; \theta\right],
\end{aligned}
$$

by Lemma 2. If $s > 1$, we again rename $\delta = (\delta_3, \dots, \delta_s)$ and $\epsilon = (\epsilon_3, \dots, \epsilon_s)$ and write

$$
\begin{aligned}
L = \sum_{\delta_1\in I} (-\theta')^{\bar{\delta}_1} &\sum_{\delta,\epsilon\in I^{s-2}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} \\
&\times \sum_{\delta_2,\epsilon_2\in I} (-\theta)^{\bar{\delta}_2\bar{\epsilon}_2}(1-q)^{\delta_2\epsilon_2} f\left[x^{a_1-\delta_1} y^{b_1} \prod_{i=2}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i}; 1, \{0\}^{s-1}; \theta\right].
\end{aligned}
$$

We now apply Lemma 4, first with $j = 2$, and again with $j = 3$, and so on up to $j = s$. The result is that

$$
L = \sum_{\substack{\delta=(\delta_1,\dots,\delta_s)\in I^s \\ \epsilon=(\epsilon_1,\dots,\epsilon_s)\in I^s \\ \epsilon_1=0}} (-\theta')^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} f\left[\left(\prod_{i=1}^{s-1} x^{a_i-\delta_i} y^{b_i-\epsilon_{i+1}}\right) x^{a_s-\delta_s} y^{b_s}; \{1\}^s; \theta\right]. \tag{3.3}
$$

On the other hand, if $s = 1$, we have (3.3) with no application of Lemma 4. In any case, applying Lemma 5 to (3.3) yields

$$
L = \sum_{\substack{\delta,\epsilon\in I^s \\ \epsilon_1=0}} (-\theta')^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} \sum_{\epsilon_{s+1}\in I} (-\theta')^{\bar{\epsilon}_{s+1}-1} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_{i+1}}; \{0\}^s; \theta'\right].
$$

If we now extend δ and ϵ by adjoining an extra component to each, viz. $\delta_{s+1} = 0$ and $\epsilon_{s+1} \in I$ respectively, we find that

$$
L = \sum_{\substack{\delta,\epsilon\in I^{s+1} \\ \delta_{s+1}=\epsilon_1=0}} (-\theta')^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_{i+1}}; \{0\}^s; \theta'\right],
$$

as required.

The proof in the case $a_1 = 1$, $b_s > 1$ is similar. The main difference is that $\delta_1 = 0$ and we begin by applying Lemma 3 instead of Lemma 2. For purposes of brevity, we suppress the details.

It is convenient to split the case $a_1 > 1$, $b_s = 1$ into the two subcases $s > 1$ and $s = 1$, since in the former we end by applying Lemma 6, while in the latter we instead use Lemma 7. Suppose first that $s > 1$. We have

$$L = \sum_{\substack{\delta,\epsilon\in I^s\\ \epsilon_s=0}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} f\left[\prod_{i=1}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i}; \{0\}^s; \theta\right]$$
$$= \sum_{\substack{\delta,\epsilon\in I^{s-1}\\ \epsilon_s=0}} (-\theta)^{\bar{\delta}\cdot\bar{\epsilon}}(1-q)^{\delta\cdot\epsilon} \sum_{\delta_1,\epsilon_1\in I} f\left[x^{a_1-\delta_1} y^{b_1-\epsilon_1}\prod_{i=2}^{s} x^{a_i-\delta_i} y^{b_i-\epsilon_i}; \{0\}^s; \theta\right].$$

Now apply Lemma 2, and then Lemma 4 successively, with $j = 2, 3, \ldots, s-1$. The result is

$$L = \sum_{\substack{\eta,\nu\in I^{s-1}\\ \nu_1=0}} (-\theta')^{\bar{\eta}\cdot\bar{\nu}}(1-q)^{\eta\cdot\nu} \sum_{\substack{\delta_s\in I\\ \nu_s=0}} (-\theta)^{\bar{\delta}_s}$$
$$\times f\left[\left(\prod_{i=1}^{s-1} x^{a_i-\eta_i} y^{b_i-\nu_{i+1}}\right) x^{a_s-\delta_s} y; \{1\}^{s-1}, 0; \theta\right].$$

Lemma 6 now gives

$$L = \sum_{\substack{\eta,\nu\in I^s\\ \nu_1=0}} (-\theta')^{\bar{\eta}\cdot\bar{\nu}}(1-q)^{\eta\cdot\nu} f\left[\prod_{i=1}^{s-1} x^{a_i-\eta_i} y^{b_i-\nu_{i+1}} x^{a_s-\eta_s} y; \{0\}^s; \theta'\right]$$
$$= \sum_{\substack{\eta,\nu\in I^{s+1}\\ \eta_{s+1}=\nu_1=\nu_{s+1}=0}} (-\theta')^{\bar{\eta}\cdot\bar{\nu}-1}(1-q)^{\eta\cdot\nu} f\left[\prod_{i=1}^{s} x^{a_i-\eta_i} y^{b_i-\nu_{i+1}}; \theta'\right],$$

as required. On the other hand, if $s = 1$ note that in this case Theorem 7 is just a restatement of Lemma 7.

The final case, with $a_1 = b_s = 1$ and $s > 1$, is proved in much the same way as the other cases with $s > 1$. Observe that now $\delta_1 = \epsilon_s = 0$ in the sum on the left, and $\delta_1 = \epsilon_{s+1} = 0$ on the right. The result is established by applying Lemma 3, then Lemma 4 successively as necessary for $j = 2, 3, \ldots, s-1$, and finally Lemma 6.

Thus, f satisfies the difference equation as claimed. This and the fact that $g[w;\theta] = f[\tau(w);\theta]$ readily implies that g satisfies the same difference equation. □

3.2. Proofs of Lemmas 1–8

We begin with the proof of Lemma 1.

Proof of Lemma 1. From the penultimate step in (3.2), noting that $n = B_s$, we have

$$f[w;\theta] = \sum_{m_1>\cdots>m_{B_s}>0} \prod_{j=1}^{B_s} \frac{q^{(k_j-1)m_j}}{[m_j]_q^{k_j-1}([m_j]_q - \theta q^{m_j})}$$
$$= \sum_{m_1>\cdots>m_{B_s}>0} \sum_{k=1}^{B_s} \frac{E_k[w;m_1,\ldots,m_{B_s}]}{[m_k]_q - \theta q^{m_k}},$$

where the partial fraction decomposition

$$\sum_{h=1}^{B_s} \frac{E_h[w;m_1,\ldots,m_{B_s}]}{[m_h]_q - \theta q^{m_h}} = \prod_{j=1}^{B_s} \frac{q^{(k_j-1)m_j}}{[m_j]_q^{k_j-1}([m_j]_q - \theta q^{m_j})}$$

implies that

$$\sum_{h=1}^{B_s} E_h[w;m_1,\ldots,m_{B_s}] \prod_{\substack{j=1\\ j\neq h}}^{B_s} \bigl([m_j]_q - \theta q^{m_j}\bigr) = \prod_{j=1}^{B_s} \frac{q^{(k_j-1)m_j}}{[m_j]_q^{k_j-1}}.$$

Letting $\theta \to q^{-m_k}[m_k]_q$ now gives that

$$E_k[w;m_1,\ldots,m_{B_s}] = \left\{ \prod_{j=1}^{B_s} \frac{q^{(k_j-1)m_j}}{[m_j]_q^{k_j-1}} \right\} \Bigg/ \prod_{\substack{j=1\\ j\neq k}}^{B_s} \bigl([m_j]_q - q^{m_j-m_k}[m_k]_q\bigr).$$

The general formula for $E_k[m_1,\ldots,m_{k-1},\nu,m_{k+1},\ldots,m_{B_s}]$ now follows immediately on replacing m_k by ν and noting that $k_j = a_i + 1$ precisely when $j = 1 + B_{i-1}$; otherwise $k_j = 1$. The lemma itself now follows on interchanging order of summation. $\square$

Proofs of several of the remaining lemmata make use of the partial fraction identity

$$\frac{\theta q^{2m}}{[m]_q^a([m]_q - \theta q^m)} - \frac{q^m}{[m]_q^{a-1}([m]_q - \theta q^m)}$$
$$= \frac{\theta' q^{2m-a}}{[m-1]_q^a([m]_q - \theta q^m)} - \frac{q^{m-a+1}}{[m-1]_q^{a-1}([m]_q - \theta q^m)}$$
$$+ \frac{q^{m-a+1}}{[m-1]_q^a} - \frac{q^m}{[m]_q^a}, \tag{3.4}$$

valid for $a > 0$ and $m > 1$.

Proof of Lemma 2. Let

$$B := \prod_{j=2}^{n} \frac{q^{(k_j-1)m_j}}{[m_j]_q^{k_j-1}([m_j]_q - \theta q^{m_j})}. \tag{3.5}$$

Then by (3.4),

$$
\begin{aligned}
&\theta f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \{0\}^s; \theta\right] - f\left[x^{a_1-1} y^{b_1} \prod_{i=2}^{s} x^{a_i} y^{b_i}; \{0\}^s; \theta\right] \\
&\quad = \sum_{m_1>\cdots>m_n>0} \left\{\frac{\theta q^{2m_1}}{[m_1]_q^{a_1}([m_1]_q - \theta q^{m_1})} - \frac{q^{m_1}}{[m_1]_q^{a_1-1}([m_1]_q - \theta q^{m_1})}\right\} q^{(a_1-2)m_1} B \\
&\quad = \sum_{m_1>\cdots>m_n>0} \left\{\frac{\theta' q^{2m_1-a_1}}{[m_1-1]_q^{a_1}([m_1]_q - \theta q^{m_1})} - \frac{q^{m_1-a_1+1}}{[m_1-1]_q^{a_1-1}([m_1]_q - \theta q^{m_1})}\right. \\
&\qquad \left. + \frac{q^{m_1-a_1+1}}{[m_1-1]_q^{a_1}} - \frac{q^{m_1}}{[m_1]_q^{a_1}}\right\} q^{(a_1-2)m_1} B \\
&\quad = \sum_{m_1>\cdots>m_n>0} \left\{\frac{\theta' q^{a_1(m_1-1)}}{[m_1-1]_q^{a_1}([m_1]_q - \theta q^{m_1})} - \frac{q^{(a_1-1)(m_1-1)}}{[m_1-1]_q^{a_1-1}([m_1]_q - \theta q^{m_1})}\right\} B \\
&\qquad + \sum_{m_2>\cdots>m_n>0} B \sum_{m_1=m_2+1}^{\infty} \left\{\frac{q^{(a_1-1)(m_1-1)}}{[m_1-1]_q^{a_1}} - \frac{q^{(a_1-1)m_1}}{[m_1]_q^{a_1}}\right\} \\
&\quad = \theta' f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; 1; \{0\}^{s-1} \theta\right] - f\left[x^{a_1-1} y^{b_1} \prod_{i=2}^{s} x^{a_i} y^{b_i}; 1; \{0\}^{s-1}; \theta\right] \\
&\qquad + \sum_{m_2>\cdots>m_n>0} \frac{B q^{(a_1-1)m_2}}{[m_2]_q^{a_1}}.
\end{aligned}
$$

But

$$
\begin{aligned}
\sum_{m_2>\cdots>m_n>0} \frac{B q^{(a_1-1)m_2}}{[m_2]_q^{a_1}} &= \sum_{m_2>\cdots>m_n>0} \left\{\frac{q^{a_1 m_2}}{[m_2]_q^{a_1}} + (1-q)\frac{q^{(a_1-1)m_2}}{[m_2]_q^{a_1-1}}\right\} B \\
&= f\left[x^{a_1} y^{b_1-1} \prod_{i=2}^{s} x^{a_i} y^{b_i}; \{0\}^s; \theta\right] \\
&\quad + (1-q) f\left[x^{a_1-1} y^{b_1-1} \prod_{i=2}^{s} x^{a_i} y^{b_i}; \{0\}^s; \theta\right],
\end{aligned}
$$

and the result follows. □

Proof of Lemma 3. Again, let B be given by (3.5). In this case (3.4) gives

$$
\theta f\left[x y^{b_1} \prod_{i=2}^{s} x^{a_i} y^{b_i}; \{0\}^s; \theta\right]
$$

$$
\begin{aligned}
&= \sum_{m_1>\cdots>m_n>0} \frac{\theta q^{2m_1}}{[m_1]_q([m_1]_q-\theta q^{m_1})}\cdot q^{-m_1}B\\
&= \sum_{m_1>\cdots>m_n>0} \left\{\frac{\theta' q^{2m_1-1}}{[m_1-1]_q([m_1]_q-\theta q^{m_1})}+\frac{q^{m_1}}{[m_1-1]_q}-\frac{q^{m_1}}{[m_1]_q}\right\}q^{-m_1}B\\
&= \sum_{m_1>\cdots>m_n>0} \frac{\theta' q^{m_1-1}B}{[m_1-1]_q([m_1]-\theta q^{m_1})}\\
&\quad + \sum_{m_2>\cdots>m_n>0} B\sum_{m_1=m_2+1}^{\infty}\left(\frac{1}{[m_1-1]_q}-\frac{1}{[m_1]_q}\right)\\
&= \theta' f\left[xy^{b_1}\prod_{i=2}^{s}x^{a_i}y^{b_i};1,\{0\}^{s-1};\theta\right]+\sum_{m_2>\cdots>m_n>0}\left(\frac{1}{[m_2]_q}+q-1\right)B.
\end{aligned}
$$

In light of $q-1+1/[m_2]_q = q^{m_2}/[m_2]_q$, it follows that

$$
\begin{aligned}
&\theta f\left[xy^{b_1}\prod_{i=2}^{s}x^{a_i}y^{b_i};\{0\}^s;\theta\right]-\theta' f\left[xy^{b_1}\prod_{i=2}^{s}x^{a_i}y^{b_i};1,\{0\}^{s-1};\theta\right]\\
&\quad = \sum_{m_2>\cdots>m_n>0}\frac{Bq^{m_2}}{[m_2]_q} = f\left[xy^{b_1-1}\prod_{i=2}^{s}x^{a_i}y^{b_i};\{0\}^s;\theta\right],
\end{aligned}
$$

as claimed. □

Proof of Lemma 4. Let $m = m_{(1+B_{j-1})}$. Define the quantities A and B by

$$
A = \prod_{i=1}^{j-1}\frac{q^{a_i(m_{(1+B_{i-1})}-d_i)}}{[m_{(1+B_{i-1})}-d_i]_q^{a_i}}\prod_{h=1+B_{i-1}}^{B_i}\frac{1}{[m_h]_q-\theta q^{m_h}},
$$

and

$$
\frac{q^{a_j m}B}{[m]_q^{a_j}([m]_q-\theta q^m)} = \prod_{i=j}^{s}\frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}\prod_{h=1+B_{i-1}}^{B_i}\frac{1}{[m_h]_q-\theta q^{m_h}}.
$$

Then (3.4) gives

$$
\begin{aligned}
&\theta f\left[\prod_{i=1}^{s}x^{a_i}y^{b_i};\{1\}^{j-1},\{0\}^{s-j+1};\theta\right]\\
&\quad - f\left[\left(\prod_{i=1}^{j-1}x^{a_i}y^{b_i}\right)x^{a_j-1}y^{b_j}\prod_{i=j+1}^{s}x^{a_i}y^{b_i};\{1\}^{j-1},\{0\}^{s-j+1};\theta\right]
\end{aligned}
$$

$$= \sum_{m_1>\cdots>m_n>0} A\left\{\frac{\theta q^{2m}}{[m]_q^{a_j}([m]_q-\theta q^m)} - \frac{q^m}{[m]_q^{a_j-1}([m]_q-\theta q^m)}\right\} q^{(a_j-2)m} B$$

$$= \sum_{m_1>\cdots>m_n>0} A\left\{\frac{\theta' q^{2m-a_j}}{[m-1]_q^{a_j}([m]_q^{a_j}-\theta q^m)} - \frac{q^{m-a_j+1}}{[m-1]_q^{a_j-1}([m]_q-\theta q^m)}\right.$$

$$\left. + \frac{q^{m-a_j+1}}{[m-1]_q^{a_j}} - \frac{q^m}{[m]_q^{a_j}}\right\} q^{(a_j-2)m} B$$

$$= \sum_{m_1>\cdots>m_n>0} A\left\{\frac{\theta' q^{a_j(m-1)}}{[m-1]_q^{a_j}([m]_q-\theta q^m)} - \frac{q^{(a_j-1)(m-1)}}{[m-1]_q^{a_j-1}([m]_q-\theta q^m)}\right\} B$$

$$+ \sum_{m_1>\cdots>m_n>0} A\left\{\frac{q^{(a_j-1)(m-1)}}{[m-1]_q^{a_j}} - \frac{q^{(a_j-1)m}}{[m]_q^{a_j}}\right\} B$$

$$= \theta' f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \{1\}^j, \{0\}^{s-j}; \theta\right]$$

$$- f\left[\left(\prod_{i=1}^{j-1} x^{a_i} y^{b_i}\right) x^{a_j-1} y^{b_j} \prod_{i=j+1}^{s} x^{a_i} y^{b_i}; \{1\}^j, \{0\}^{s-j}; \theta\right]$$

$$+ \sum_{\substack{m_1>\cdots>m_{B_{j-1}} \\ -1+m_{B_{j-1}}>m_{(2+B_{j-1})}>\cdots>m_{B_s}>0}} AB \sum_{m=1+m_{(2+B_{j-1})}}^{-1+m_{B_{j-1}}} \left(\frac{q^{(a_j-1)(m-1)}}{[m-1]_q^{a_j}} - \frac{q^{(a_j-1)m}}{[m]_q^{a_j}}\right).$$

It follows that

$$\theta f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \{1\}^{j-1}, \{0\}^{s-j+1}; \theta\right]$$

$$- f\left[\left(\prod_{i=1}^{j-1} x^{a_i} y^{b_i}\right) x^{a_j-1} y^{b_j} \prod_{i=j+1}^{s} x^{a_i} y^{b_i}; \{1\}^{j-1}, \{0\}^{s-j+1}; \theta\right]$$

$$- \theta' f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \{1\}^j, \{0\}^{s-j}; \theta\right]$$

$$+ f\left[\left(\prod_{i=1}^{j-1} x^{a_i} y^{b_i}\right) x^{a_j-1} y^{b_j} \prod_{i=j+1}^{s} x^{a_i} y^{b_i}; \{1\}^j, \{0\}^{s-j}; \theta\right]$$

$$= \sum_{\substack{m_1>\cdots>m_{B_{j-1}} \\ -1+m_{B_{j-1}}>m_{(2+B_{j-1})}>\cdots>m_{B_s}>0}} A\left\{\frac{q^{(a_j-1)m_{(2+B_{j-1})}}}{[m_{(2+B_{j-1})}]_q^{a_j}} - \frac{q^{(a_j-1)(-1+m_{B_{j-1}})}}{[-1+m_{B_{j-1}}]_q^{a_j}}\right\} B$$

$$= \sum_{m_1 > \cdots > m_{B_{j-1}} > m_{(2+B_{j-1})} > \cdots > m_{B_s} > 0} A\left\{ \frac{q^{a_j m_{(2+B_{j-1})}}}{[m_{(2+B_{j-1})}]_q^{a_j}} + (1-q)\frac{q^{(a_j-1)m_{(2+B_{j-1})}}}{[m_{(2+B_{j-1})}]_q^{a_j-1}} \right.$$

$$\left. - \frac{q^{a_j(-1+m_{B_{j-1}})}}{[-1+m_{B_{j-1}}]_q^{a_j}} - (1-q)\frac{q^{(a_j-1)(-1+m_{B_{j-1}})}}{[-1+m_{B_{j-1}}]_q^{a_j-1}} \right\} B$$

$$= f\left[\left(\prod_{i=1}^{j-1} x^{a_i} y^{b_i} \right) x^{a_j} y^{b_j-1} \prod_{i=j+1}^{s} x^{a_i} y^{b_i}; \{1\}^{j-1}, \{0\}^{s-j+1}; \theta \right]$$

$$+ (1-q) f\left[\left(\prod_{i=1}^{j-1} x^{a_i} y^{b_i} \right) x^{a_j-1} y^{b_j-1} \prod_{i=j+1}^{s} x^{a_i} y^{b_i}; \{1\}^{j-1}, \{0\}^{s-j+1}; \theta \right]$$

$$- f\left[\left(\prod_{i=1}^{j-2} x^{a_i} y^{b_i} \right) x^{a_{j-1}} y^{b_{j-1}-1} \prod_{i=j}^{s} x^{a_i} y^{b_i}; \{1\}^{j}, \{0\}^{s-j}; \theta \right]$$

$$- (1-q) f\left[\left(\prod_{i=1}^{j-2} x^{a_j} y^{b_j} \right) x^{a_{j-1}} y^{b_{j-1}-1} x^{a_j-1} y^{b_j} \prod_{i=j+1}^{s} x^{a_i} y^{b_i}; \{1\}^{j}, \{0\}^{s-j}; \theta \right],$$

as required. □

Proof of Lemma 5. Here $b_s > 1$, and thus if we shift summation indices $m_i \mapsto 1 + m_i$, then

$$f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \{1\}^s; \theta \right]$$

$$= \sum_{m_1 > \cdots > m_{B_s} > 0} \prod_{i=1}^{s} \frac{q^{a_i(m_{(1+B_{i-1})}-1)}}{[m_{(1+B_{i-1})}-1]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta q^{m_j}}$$

$$= \sum_{m_1 > \cdots > m_{B_s} \geqslant 0} \prod_{i=1}^{s} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j+1]_q - \theta q^{m_j+1}}$$

$$= \sum_{m_1 > \cdots > m_{B_s} \geqslant 0} \prod_{i=1}^{s} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta' q^{m_j}}$$

$$= \left(\sum_{m_1 > \cdots > m_{B_s} > 0} + \sum_{m_1 > \cdots > m_{(B_s-1)} > 0} \right) \prod_{i=1}^{s} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}$$

$$\times \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta' q^{m_j}}$$

$$= f\left[\prod_{i=1}^{s} x^{a_i} y^{b_i}; \{0\}^s; \theta'\right] - \left(\frac{1}{\theta'}\right) f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s} y^{b_s-1}; \{0\}^s; \theta'\right]. \qquad \square$$

Proof of Lemma 6. In this case, $B_s = 1 + B_{s-1}$ and we have

$$f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s-1} y; \{1\}^{s-1}, 0; \theta\right] - \theta f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s} y; \{1\}^{s-1}, 0; \theta\right]$$
$$= \sum_{m_1>\cdots>m_{B_s}>0} \left\{\prod_{i=1}^{s-1} \frac{q^{a_i(m_{(1+B_{i-1})}-1)}}{[m_{(1+B_{i-1})}-1]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta q^{m_j}}\right\}$$
$$\times \left\{\frac{q^{(a_s-1)m_{B_s}}}{[m_{B_s}]_q^{a_s-1}} - \frac{\theta q^{a_s m_{B_s}}}{[m_{B_s}]_q^{a_s}}\right\} \frac{1}{[m_{B_s}]_q - \theta q^{m_{B_s}}}$$
$$= \sum_{m_1>\cdots>m_{B_s}>0} \left\{\prod_{i=1}^{s-1} \frac{q^{a_i(m_{(1+B_{i-1})}-1)}}{[m_{(1+B_{i-1})}-1]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta q^{m_j}}\right\} \frac{q^{(a_s-1)m_{B_s}}}{[m_{B_s}]_q^{a_s}}.$$

Now shift the summation indices $m_i \mapsto m_i + 1$ and use $[m+1]_q - \theta q^{m+1} = [m]_q - \theta' q^m$ to obtain

$$f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s-1} y; \{1\}^{s-1}, 0; \theta\right] - \theta f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s} y; \{1\}^{s-1}, 0; \theta\right]$$
$$= \sum_{m_1>\cdots>m_{B_s}\geqslant 0} \left\{\prod_{i=1}^{s-1} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta' q^{m_j}}\right\} \frac{q^{(a_s-1)m_{1+B_s}}}{[1+m_{B_s}]_q^{a_s}}.$$

Now replace m_{B_s} by $m_{B_s} - 1$. Then

$$f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s-1} y; \{1\}^{s-1}, 0; \theta\right] - \theta f\left[\left(\prod_{i=1}^{s-1} x^{a_i} y^{b_i}\right) x^{a_s} y; \{1\}^{s-1}, 0; \theta\right]$$
$$= \sum_{m_1>\cdots>m_{B_s-1}\geqslant m_{B_s}>0} \left\{\prod_{i=1}^{s-1} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta' q^{m_j}}\right\} \frac{q^{(a_s-1)m_{B_s}}}{[m_{B_s}]_q^{a_s}}$$
$$= \sum_{m_1>\cdots>m_{B_s-1}>m_{B_s}>0} \left\{\prod_{i=1}^{s-1} \frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}} \prod_{j=1+B_{i-1}}^{B_i} \frac{1}{[m_j]_q - \theta' q^{m_j}}\right\}$$
$$\times \frac{q^{(a_s-1)m_{B_s}}}{[m_{B_s}]_q^{a_s}} \cdot \frac{[m_{B_s}]_q - \theta' q^{m_{B_s}}}{[m_{B_s}]_q - \theta' q^{m_{B_s}}}$$

$$
\begin{aligned}
&+\sum_{m_1>\cdots>m_{B_s-1}>0}\left\{\prod_{i=1}^{s-1}\frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}\prod_{j=1+B_{i-1}}^{B_i}\frac{1}{[m_j]_q-\theta' q^{m_j}}\right\}\frac{q^{(a_s-1)m_{(B_s-1)}}}{[m_{(B_s-1)}]_q^{a_s}}\\
&=\sum_{m_1>\cdots>m_{B_s}>0}\left\{\prod_{i=1}^{s-1}\frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}\prod_{j=1+B_{i-1}}^{B_i}\frac{1}{[m_j]_q-\theta' q^{m_j}}\right\}\\
&\quad\times\frac{q^{(a_s-1)m_{B_s}}}{[m_{B_s}]_q^{a_s-1}([m_{B_s}]_q-\theta' q^{m_{B_s}})}\\
&\quad-\theta'\sum_{m_1>\cdots>m_{B_s}>0}\left\{\prod_{i=1}^{s-1}\frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}\prod_{j=1+B_{i-1}}^{B_i}\frac{1}{[m_j]_q-\theta' q^{m_j}}\right\}\\
&\quad\times\frac{q^{a_s m_{B_s}}}{[m_{B_s}]_q^{a_s}([m_{B_s}]_q-\theta' q^{m_{B_s}})}\\
&\quad+\sum_{m_1>\cdots>m_{B_{s-1}}>0}\left\{\prod_{i=1}^{s-1}\frac{q^{a_i m_{(1+B_{i-1})}}}{[m_{(1+B_{i-1})}]_q^{a_i}}\prod_{j=1+B_{i-1}}^{B_i}\frac{1}{[m_j]_q-\theta' q^{m_j}}\right\}\frac{q^{(a_s-1)m_{B_{s-1}}}}{[m_{B_{s-1}}]_q^{a_s}}\\
&=f\left[\left(\prod_{i=1}^{s-1}x^{a_i}y^{b_i}\right)x^{a_s-1}y;\{0\}^s;\theta'\right]-\theta' f\left[\left(\prod_{i=1}^{s-1}x^{a_i}y^{b_i}\right)x^{a_s}y;\{0\}^s;\theta'\right]\\
&\quad+f\left[\left(\prod_{i=1}^{s-2}x^{a_i}y^{b_i}\right)x^{a_{s-1}}y^{b_{s-1}-1}x^{a_s}y;\{0\}^s;\theta'\right]\\
&\quad+(1-q)f\left[\left(\prod_{i=1}^{s-2}x^{a_i}y^{b_i}\right)x^{a_{s-1}}y^{b_{s-1}-1}x^{a_s-1}y;\{0\}^s;\theta'\right].\qquad\square
\end{aligned}
$$

Proof of Lemma 7. If $a>1$, then

$$
\begin{aligned}
f\big[x^{a-1}y;\theta\big]-\theta f\big[x^a y;\theta\big]&=\sum_{m=1}^{\infty}\left(\frac{q^{(a-1)m}}{[m]_q^{a-1}}-\frac{\theta q^{am}}{[m]_q^a}\right)\frac{1}{[m]_q-\theta q^m}\\
&=\sum_{m=1}^{\infty}\frac{q^{(a-1)m}}{[m]_q^a}=\sum_{m=1}^{\infty}\frac{q^{(a-1)m}}{[m]_q^a}\cdot\frac{[m]_q-\theta' q^m}{[m]_q-\theta' q^m}\\
&=\sum_{m=1}^{\infty}\frac{q^{(a-1)m}}{[m]_q^{a-1}([m]_q-\theta' q^m)}-\theta'\sum_{m=1}^{\infty}\frac{q^{am}}{[m]_q^a([m]_q-\theta' q^m)}\\
&=f\big[x^{a-1}y;\theta'\big]-\theta' f\big[x^a y;\theta'\big].\qquad\square
\end{aligned}
$$

Proof of Lemma 8. Let n be a positive integer. Then

$$
\begin{aligned}
&\theta \sum_{m=1}^{n} \frac{q^m}{[m]_q([m]_q - \theta q^m)} - \theta' \sum_{m=1}^{n} \frac{q^m}{[m]_q([m]_q - \theta q^m)} \\
&\quad = \sum_{m=1}^{n} \frac{[m]_q - \theta' q^m}{[m]_q([m]_q - \theta' q^m)} - \sum_{m=1}^{n} \frac{1}{[m]_q - \theta' q^m} \\
&\qquad - \sum_{m=1}^{n} \frac{[m]_q - \theta q^m}{[m]_q([m]_q - \theta q^m)} + \sum_{m=1}^{n} \frac{1}{[m]_q - \theta q^m} \\
&\quad = \sum_{m=1}^{n} \frac{1}{[m]_q - \theta q^m} - \sum_{m=1}^{n} \frac{1}{[m]_q - \theta' q^m} \\
&\quad = \sum_{m=0}^{n-1} \frac{1}{[m+1]_q - \theta q^{m+1}} - \sum_{m=1}^{n} \frac{1}{[m]_q - \theta' q^m} \\
&\quad = \sum_{m=0}^{n-1} \frac{1}{[m]_q - \theta' q^m} - \sum_{m=1}^{n} \frac{1}{[m]_q - \theta' q^m} \\
&\quad = \frac{1}{-\theta'} - \frac{1}{[n]_q - \theta' q^n}.
\end{aligned}
$$

The result now follows on letting $n \to \infty$. $\square$

4. Derivations

We continue to employ the algebraic notation of the previous section, and write $\hat{\zeta}(\cdot)$ for the $q = 1$ case of the $\mathbf{Q}$-linear map $\hat{\zeta}[\cdot]$ defined there. Thus, $\hat{\zeta}(x^{s_1-1}y \cdots x^{s_m-1}y) = \zeta(s_1, \ldots, s_m)$ gives the ordinary multiple zeta value. Note that q-duality (Corollary 3) simply says that $\hat{\zeta}[\tau w] = \hat{\zeta}[w]$ for all words $w \in \mathfrak{h}^0$, while ordinary duality reduces to $\hat{\zeta}(\tau w) = \hat{\zeta}(w)$. In contrast [10], for (1.3) the relevant algebra is not $\mathfrak{h}^0$, but $\mathfrak{h}y$, with the automorphism $w \mapsto (Jw)x^{-1}y$ (where J switches x and y but preserves the order of the word) replacing the anti-automorphism τ.

If D is a derivation of $\mathfrak{h}$, let $\overline{D}$ denote the conjugate derivation $\tau D \tau$. As in [19], we refer to D as symmetric (respectively antisymmetric) if $\overline{D} = D$ ($\overline{D} = -D$), and note that any symmetric or antisymmetric derivation is completely determined by where it sends x. Ihara and Kaneko [20] defined a family of antisymmetric derivations ∂_n for positive integers n by declaring that $\partial_n(x) = x(x+y)^{n-1}y$. They conjectured—and subsequently proved—that for all positive integers n and words $w \in \mathfrak{h}^0$, $\hat{\zeta}(\partial_n(w)) = 0$. Here, we shall prove that this result extends to the multiple q-zeta function.

Theorem 8. *For all positive integers n and words $w \in \mathfrak{h}^0$, $\hat{\zeta}[\partial_n(w)] = 0$.*

Proof. Again, for positive integer n let D_n be the derivation mapping $x \mapsto 0$ and $y \mapsto x^n y$. Fix a formal power series parameter t and set

$$D := \sum_{n=1}^{\infty} t^n \frac{D_n}{n}, \qquad \sigma := \exp(D), \qquad \partial := \sum_{n=1}^{\infty} t^n \frac{\partial_n}{n}.$$

The reformulated version of the generalized q-duality theorem (Theorem 6) states that $\hat{\zeta}[\sigma w] = \hat{\zeta}[\sigma\tau w]$ for all $w \in \mathfrak{h}^0$. In view of the special case, q-duality (Corollary 3), this is equivalent to $(\sigma - \bar{\sigma})w \in \ker\hat{\zeta}$ for all $w \in \mathfrak{h}^0$. We show that in fact, $(\sigma - \bar{\sigma})\mathfrak{h}^0 = \partial\mathfrak{h}^0$, from which it follows that Theorem 8 is equivalent to generalized q-duality. To prove the equivalence, we require the following identity of Ihara and Kaneko [20].

Proposition 1 [19, Theorem 5.9]. *We have the following equality of* $\mathfrak{h}[\![t]\!]$ *automorphisms*: $\exp(\partial) = \bar{\sigma}\sigma^{-1}$.

To complete the proof of Theorem 8, observe as in [20,30] that since

$$\partial = \log\bigl(\bar{\sigma}\sigma^{-1}\bigr) = \log\bigl(1 - (\sigma - \bar{\sigma})\sigma^{-1}\bigr) = -(\sigma - \bar{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n}\bigl((\sigma - \bar{\sigma})\sigma^{-1}\bigr)^{n-1}\sigma^{-1},$$

and

$$\sigma - \bar{\sigma} = \bigl(1 - \bar{\sigma}\sigma^{-1}\bigr)\sigma = \bigl(1 - \exp(\partial)\bigr)\sigma = -\partial \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{n!}\sigma,$$

we see that $\partial\mathfrak{h}^0 \subseteq (\sigma - \bar{\sigma})\mathfrak{h}^0$ and $(\sigma - \bar{\sigma})\mathfrak{h}^0 \subseteq \partial\mathfrak{h}^0$. Thus for the kernel of $\hat{\zeta}$, we have the equivalences

$$(\sigma - \bar{\sigma})w \in \ker\hat{\zeta} \iff \partial w \in \ker\hat{\zeta} \iff \forall n \in \mathbf{Z}^+,\ \hat{\zeta}[\partial_n w] = 0. \qquad \square$$

Remark 3. The proof of Proposition 1 that is given in [19] involves imposing a Hopf algebra structure on $\mathfrak{h}$ and defining an action on it. Zudilin [30, Lemma 7] presents an alternative proof in the case $t = 1$ along the lines originally indicated by Ihara and Kaneko [20]. It is possible to extend Zudilin's presentation [30] to arbitrary t by defining a family $\{\varphi_s\colon\ s \in \mathbf{R}\}$ of automorphisms of $\mathbf{R}\langle\langle x, y\rangle\rangle$ defined on the generators $z = x + y$ and y by

$$\varphi_s(z) = z, \qquad \varphi_s(y) = (1 - tz)^s y\left(1 - \frac{1 - (1 - tz)^s}{z} y\right)^{-1}.$$

Routine calculations on the generators verify the equalities

$$\varphi_{s_1} \circ \varphi_{s_2} = \varphi_{s_1+s_2}, \qquad \varphi_0 = \mathrm{id}, \qquad \frac{d}{ds}\varphi_s\bigg|_{s=0} = \partial, \qquad \varphi_1 = \bar{\sigma}\sigma^{-1}.$$

The first three results imply that $\varphi_s = \exp(s\partial)$, and the substitution $s = 1$ gives Proposition 1.

Remark 4. In view of the identity $\partial_1 = \overline{D}_1 - D_1$, the case $n = 1$ of Theorem 8 yields the following q-analog of Hoffman's derivation theorem [16, Theorem 5.1], [19, Theorem 2.1]:

Corollary 5. *For any word $w \in \mathfrak{h}^0$, $\hat{\zeta}[D_1 w] = \hat{\zeta}[\overline{D}_1 w]$. Equivalently, if $s_1, \dots, s_m$ are positive integers with $s_1 > 1$, then*

$$\sum_{k=1}^{m} \zeta\left[\operatorname*{Cat}_{j=1}^{k-1} s_j, 1 + s_k, \operatorname*{Cat}_{j=k+1}^{m} s_j\right] = \sum_{k=1}^{m} \sum_{j=0}^{s_k - 2} \zeta\left[\operatorname*{Cat}_{i=1}^{k-1} s_i, s_k - j, j + 1, \operatorname*{Cat}_{i=k+1}^{m} s_i\right].$$

By the usual convention on empty sums, the sum on the right is zero if $s_k < 2$.

5. Cyclic sums

In this section, we state and prove a q-analog of the cyclic sum theorem [19], originally conjectured by Hoffman and subsequently proved by Ohno using a partial fractions argument. As a corollary, we give another proof of the q-sum formula (Corollary 4).

Theorem 9 (q-Cyclic sum formula). *Let n and $s_1, s_2, \dots, s_n$ be positive integers such that $s_j > 1$ for some j. Then*

$$\sum_{j=1}^{n} \zeta\left[s_j + 1, \operatorname*{Cat}_{m=j+1}^{n} s_m, \operatorname*{Cat}_{m=1}^{j-1} s_m\right] = \sum_{j=1}^{n} \sum_{k=0}^{s_j - 2} \zeta\left[s_j - k, \operatorname*{Cat}_{m=j+1}^{n} s_m, \operatorname*{Cat}_{m=1}^{j-1} s_m, k + 1\right].$$

Note that the inner sum on the right vanishes if $s_j = 1$. We refer to Theorem 9 as the q-cyclic sum formula because, as with the limiting case in [19], it has an elegant reformulation in terms of cyclic permutations of dual argument lists.

Definition 7. If $\vec{s} = (s_1, \dots, s_n)$ is a vector of n positive integers, let

$$\mathcal{C}(\vec{s}) = \{(s_1, \dots, s_n), (s_2, \dots, s_n, s_1), \dots, (s_n, s_1, \dots, s_{n-1})\}$$

denote the set of cyclic permutations of $\vec{s}$. Also, for notational convenience, define $\zeta^*[s_1, \dots, s_n] := \zeta[s_1 + 1, s_2, \dots, s_n]$.

We can now restate Theorem 9 as follows.

Theorem 10 (q-Analog of [19, Eq. (2)]). *Let $\mathbf{s}$ and $\mathbf{s}'$ be dual argument lists. Then*

$$\sum_{\mathbf{p} \in \mathcal{C}(\mathbf{s})} \zeta^*[\mathbf{p}] = \sum_{\mathbf{p} \in \mathcal{C}(\mathbf{s}')} \zeta^*[\mathbf{p}].$$

To prove the implication Theorem 9 $\Rightarrow$ Theorem 10, we borrow an argument of Ohno for the $q=1$ case. Let

$$\mathbf{s} = \left(\operatorname*{Cat}_{j=1}^{m}\left\{a_j+2, \{1\}^{b_j}\right\}\right) = (s_1, \dots, s_n),$$

where a_j and b_j are non-negative integers for $1 \leqslant j \leqslant m$ and $n = m + b_1 + \cdots + b_m$. The right-hand side of Theorem 9 is

$$\begin{aligned}
&\frac{|\mathcal{C}(\mathbf{s})|}{n} \sum_{\mathbf{p}\in\mathcal{C}(\mathbf{s})} \sum_{k=0}^{p_1-2} \zeta[p_1-k, p_2, \dots, p_n, k+1] \\
&\quad = \frac{|\mathcal{C}(\mathbf{s})|}{n} \sum_{(\mathbf{c},\mathbf{d})} \sum_{k=0}^{c_1} \zeta\left[c_1+2-k, \{1\}^{d_1}, \operatorname*{Cat}_{j=2}^{m}\left\{c_j+2, \{1\}^{d_j}\right\}, k+1\right],
\end{aligned}$$

where the outer sum on the right is over all cyclic permutations

$$(\mathbf{c}, \mathbf{d}) = \big((c_1, d_1), \dots, (c_m, d_m)\big)$$

of the ordered sequence of ordered pairs $((a_1, b_1), \dots, (a_m, b_m))$. Invoking q-duality (Corollary 3), we find that the right-hand side of Theorem 9 can now be expressed as

$$\begin{aligned}
&\frac{|\mathcal{C}(\mathbf{s})|}{n}\left\{\sum_{(\mathbf{c},\mathbf{d})} \zeta\left[\operatorname*{Cat}_{j=1}^{m}\left\{c_j+2, \{1\}^{d_j}\right\}, 1\right]\right. \\
&\qquad \left. + \sum_{(\mathbf{c},\mathbf{d})} \sum_{k=1}^{c_1} \zeta\left[c_1+2-k, \{1\}^{d_1}, \operatorname*{Cat}_{j=2}^{m}\left\{c_j+2, \{1\}^{d_j}\right\}, k+1\right]\right\} \\
&\quad = \frac{|\mathcal{C}(\mathbf{s})|}{n}\left\{\sum_{(\mathbf{c},\mathbf{d})} \zeta\left[d_m+3, \{1\}^{c_m}, \operatorname*{Cat}_{j=2}^{m}\left\{d_{m-j+1}+2, \{1\}^{c_{m-j+1}}\right\}\right]\right. \\
&\qquad \left. + \sum_{(\mathbf{c},\mathbf{d})} \sum_{k=1}^{c_1} \zeta\left[2, \{1\}^{k-1}, \operatorname*{Cat}_{j=1}^{m-1}\left\{d_{m-j+1}+2, \{1\}^{c_{m-j+1}}\right\}, d_1+2, \{1\}^{c_1-k}\right]\right\} \\
&\quad = \sum_{\mathbf{p}\in\mathcal{C}(\mathbf{s}')} \zeta^*[\mathbf{p}].
\end{aligned}$$

But the left-hand side of Theorem 9 is

$$\sum_{j=1}^{n} \zeta^*\left[\operatorname*{Cat}_{m=j}^{n} s_m, \operatorname*{Cat}_{m=1}^{j-1} s_m\right] = \sum_{\mathbf{p}\in\mathcal{C}(\mathbf{s})} \zeta^*[\mathbf{p}].$$

We now proceed with the proof of Theorem 9. As we shall see, much of the proof of the limiting case in [19] can be adapted to the present situation with only minor modifications. To this end, we introduce two auxiliary q-series.

Definition 8. For positive integers $s_1, s_2, \ldots, s_n$ and non-negative integer s_{n+1}, let

$$T[s_1, \ldots, s_n] := \sum_{k_1 > \cdots > k_{n+1} \geqslant 0} \frac{q^{k_1 - k_{n+1}}}{[k_1 - k_{n+1}]_q} \prod_{j=1}^{n} \frac{q^{(s_j - 1)k_j}}{[k_j]_q^{s_j}},$$

$$S[s_1, \ldots, s_{n+1}] := \sum_{k_1 > \cdots > k_{n+1} > 0} \frac{q^{k_1}}{[k_1 - k_{n+1}]_q} \prod_{j=1}^{n+1} \frac{q^{(s_j - 1)k_j}}{[k_j]_q^{s_j}}. \tag{5.1}$$

For the convergence of the q-series (5.1), we have the following generalization of [19, Theorem 3.1].

Theorem 11. *$T[s_1, \ldots, s_n]$ is finite if there is an index j with $s_j > 1$; $S[s_1, \ldots, s_{n+1}]$ is finite if one of $s_1, \ldots, s_n$ exceeds 1 or if $s_{n+1} > 0$.*

We defer the proof of Theorem 11 to the end of the section in order to proceed more directly with the proof of Theorem 9. The key result we need is a direct generalization of the corresponding result in [19]:

Theorem 12 (q-Analog of [19, Theorem 3.2]). *If $s_1, \ldots, s_n$ are positive integers with $s_j > 1$ for some j, then*

$$T[s_1, \ldots, s_n] - T[s_2, \ldots, s_n, s_1] = \zeta[s_1 + 1, s_2, \ldots, s_n] - \sum_{k=0}^{s_1 - 2} \zeta[s_1 - k, s_2, \ldots, s_n, k + 1],$$

where the sum on the right vanishes if $s_1 = 1$.

The proof of Theorem 9 now follows immediately on summing Theorem 12 over all cyclic permutations of the argument sequence $s_1, \ldots, s_n$.

Proof. Although we provide details, the argument is quite similar to the corresponding argument in [19]. One minor difference is that $\lim_{N \to \infty} 1/[N]_q = 1 - q \neq 0$ if $q \neq 1$, which affects the computations used to arrive at (5.5) below. First,

$$\begin{aligned} S[s_1, \ldots, s_n, 0] &= \sum_{k_1 > \cdots > k_{n+1} > 0} \frac{q^{k_1 - k_{n+1}}}{[k_1 - k_{n+1}]_q} \prod_{j=1}^{n} \frac{q^{(s_j - 1)k_j}}{[k_j]_q^{s_j}} \\ &= \sum_{k_1 > \cdots > k_{n+1} \geqslant 0} \frac{q^{k_1 - k_{n+1}}}{[k_1 - k_{n+1}]_q} \prod_{j=1}^{n} \frac{q^{(s_j - 1)k_j}}{[k_j]_q^{s_j}} \end{aligned}$$

$$-\sum_{k_1>\cdots>k_n>0}\frac{q^{k_1}}{[k_1]_q}\prod_{j=1}^{n}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}$$
$$=T[s_1,\dots,s_n]-\zeta[s_1+1,s_2,\dots,s_n]. \tag{5.2}$$

Next, we apply the identity

$$\frac{q^{k_1-k_{n+1}}}{[k_1-k_{n+1}]_q[k_1]_q}=\frac{1}{[k_{n+1}]_q}\left(\frac{1}{[k_1-k_{n+1}]_q}-\frac{1}{[k_1]_q}\right) \tag{5.3}$$

to $S[s_1,\dots,s_{n+1}]$. This gives

$$\sum_{k_1>\cdots>k_{n+1}>0}\frac{q^{k_1-k_{n+1}}}{[k_1-k_{n+1}]_q[k_1]_q}\cdot\frac{q^{(s_1-1)k_1}}{[k_1]_q^{s_1-1}}\cdot\frac{q^{s_{n+1}k_{n+1}}}{[k_{n+1}]_q^{s_{n+1}}}\prod_{j=2}^{n}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}$$
$$=\sum_{k_1>\cdots>k_{n+1}>0}\left(\frac{1}{[k_1-k_{n+1}]_q}-\frac{1}{[k_1]_q}\right)\frac{q^{(s_1-1)k_1}}{[k_1]_q^{s_1-1}}$$
$$\cdot\frac{q^{s_{n+1}k_{n+1}}}{[k_{n+1}]_q^{1+s_{n+1}}}\prod_{j=2}^{n}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}},$$

from which it follows that

$$S[s_1,\dots,s_{n+1}]=S[s_1-1,s_2,\dots,s_n,1+s_{n+1}]-\zeta[s_1,\dots,s_n,1+s_{n+1}]. \tag{5.4}$$

Finally, applying (5.3) to $S[1,s_2,\dots,s_n,s_{n+1}-1]$ gives

$$\sum_{k_1>\cdots>k_{n+1}>0}\frac{q^{k_1-k_{n+1}}}{[k_1-k_{n+1}]_q[k_1]_q}\cdot\frac{q^{(s_{n+1}-1)k_{n+1}}}{[k_{n+1}]_q^{s_{n+1}-1}}\prod_{j=2}^{n}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}$$
$$=\sum_{k_1>\cdots>k_{n+1}>0}\left(\frac{1}{[k_1-k_{n+1}]_q}-\frac{1}{[k_1]_q}\right)\prod_{j=2}^{n+1}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}$$
$$=\sum_{k_2>\cdots>k_{n+1}>0}\prod_{j=2}^{n+1}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}\lim_{N\to\infty}\sum_{k_1=k_2+1}^{N}\left(\frac{1}{[k_1-k_{n+1}]_q}-\frac{1}{[k_1]_q}\right)$$
$$=\sum_{k_2>\cdots>k_{n+1}>0}\prod_{j=2}^{n+1}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}\lim_{N\to\infty}\sum_{m=0}^{k_{n+1}-1}\left(\frac{1}{[k_2-m]_q}-\frac{1}{[N-m]_q}\right)$$
$$=\sum_{k_2>\cdots>k_{n+1}>0}\prod_{j=2}^{n+1}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}\sum_{m=0}^{k_{n+1}-1}\left(\frac{1}{[k_2-m]_q}+q-1\right)$$

$$= \sum_{k_2>\cdots>k_{n+1}>m\geqslant 0} \frac{q^{k_2-m}}{[k_2-m]_q} \prod_{j=2}^{n+1} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

It follows that

$$S[1, s_2, \ldots, s_n, s_{n+1}-1] = T[s_2, \ldots, s_n, s_{n+1}]. \tag{5.5}$$

Now let $0 \leqslant j \leqslant s_1 - 2$, apply (5.4) and sum on j. This yields

$$\sum_{j=0}^{s_1-2} S[s_1 - j, s_2, \ldots, s_n, j] = \sum_{j=0}^{s_1-2} \bigl(S[s_1-j-1, s_2, \ldots, s_n, j+1] - \zeta[s_1-j, s_2, \ldots, s_n, j+1]\bigr),$$

which telescopes, leaving

$$S[s_1, s_2, \ldots, s_n, 0] = S[1, s_2, \ldots, s_n, s_1-1] - \sum_{j=0}^{s_1-2} \zeta[s_1-j, s_2, \ldots, s_n, j+1].$$

Now apply (5.2) and (5.5) to obtain

$$T[s_1, \ldots, s_n] - \zeta[s_1+1, s_2, \ldots, s_n] = T[s_2, \ldots, s_n, s_1] - \sum_{j=0}^{s_1-2} \zeta[s_1-j, s_2, \ldots, s_n, j+1]. \qquad \square$$

As Ohno observed, the sum formula [15] is an easy consequence of [19, Theorem 3.2]. Correspondingly, we can give another proof of Corollary 4, our q-analog of the sum formula.

Alternative proof of Corollary 4. Sum Theorem 12 over all $s_1, \ldots, s_n$ with $s_1 + \cdots + s_n = k$. Since the resulting sum of T-functions vanishes, we get

$$\sum_{s_1+\cdots+s_n=k} \zeta[s_1+1, s_2, \ldots, s_n] = \sum_{s_1+\cdots+s_n=k} \sum_{j=0}^{s_1-2} \zeta[s_1-j, s_2, \ldots, s_n, j+1]$$
$$= \sum_{s_1+\cdots+s_{n+1}=k} \zeta[s_1+1, s_2, \ldots, s_{n+1}].$$

It follows that the sums are independent of n; whence each is equal to

$$\sum_{s_1=k} \zeta[s_1+1] = \zeta[k+1],$$

as required. $\square$

We conclude the section with a proof of Theorem 11. Again, the argument closely follows Ohno's proof of the limiting case in [19].

Proof of Theorem 11. By (5.2),

$$S[s_1,\dots,s_n,s_{n+1}] \leqslant S[s_1,\dots,s_n,0] \leqslant T[s_1,\dots,s_n],$$

so $S[s_1,\dots,s_{n+1}]$ is finite if $T[s_1,\dots,s_n]$ is. By (5.5),

$$S[1,s_2,\dots,s_n,s_{n+1}] = T[s_2,\dots,s_n,s_{n+1}+1],$$

so the statement about finiteness of S follows from the corresponding statement about T. To prove finiteness of $T[s_1,\dots,s_n]$ with $s_1+\dots+s_n>n$, it suffices to consider the case $s_1+\dots+s_n=n+1$, for if $s_k>1$, then $T[s_1,\dots,s_n]\leqslant T[\{1\}^{k-1},2,\{1\}^{n-k}]$. Thus, we need only prove that $T[\{1\}^{k-1},2,\{1\}^{n-k}]<\infty$ for $1\leqslant k\leqslant n$. When $k=1$, we have

$$\begin{aligned}
T\bigl[2,\{1\}^{n-1}\bigr] &= \sum_{k_1>\dots>k_{n+1}\geqslant 0} \frac{q^{k_1-k_{n+1}+k_1}}{[k_1-k_{n+1}]_q[k_1]_q^2}\prod_{j=2}^n \frac{1}{[k_j]_q}\\
&\leqslant \sum_{\substack{k_1>\dots>k_n>0\\ k_1\geqslant m>0}} \frac{q^{m+k_1}}{[m]_q[k_1]_q^2}\prod_{j=2}^n \frac{1}{[k_j]_q}\\
&= \zeta\bigl[3,\{1\}^{n-1}\bigr] + n\zeta\bigl[2,\{1\}^n\bigr] + \sum_{k=1}^{n-1}\zeta\bigl[2,\{1\}^{k-1},2,\{1\}^{n-k-1}\bigr]\\
&< \infty.
\end{aligned}$$

Arguing inductively, we now suppose that $T[\{1\}^{k-1},2,\{1\}^{n-k}]<\infty$ for some $k\geqslant 1$. By (5.2), (5.5) and the inductive hypothesis,

$$\begin{aligned}
T\bigl[\{1\}^k,2,\{1\}^{n-k-1}\bigr] &= S\bigl[\{1\}^k,2,\{1\}^{n-k-1},0\bigr] + \zeta\bigl[2,\{1\}^{k-1},2,\{1\}^{n-k-1}\bigr]\\
&= T\bigl[\{1\}^{k-1},2,\{1\}^{n-k}\bigr] + \zeta\bigl[2,\{1\}^{k-1},2,\{1\}^{n-k-1}\bigr]\\
&< \infty,
\end{aligned}$$

as required. □

6. Multiple q-polylogarithms

In analogy with [3, Eq. (1.1)], define

$$\lambda_q\left[\begin{matrix} s_1,\dots,s_m\\ b_1,\dots,b_m\end{matrix}\right] := \sum_{\nu_1,\dots,\nu_m>0}\prod_{k=1}^m b_k^{-\nu_k}\left[\sum_{j=k}^m \nu_j\right]_q^{-s_k}, \tag{6.1}$$

and set

$$\mathrm{Li}_{s_1,\dots,s_m}[x_1,\dots,x_m] := \sum_{n_1>\cdots>n_m>0} \prod_{k=1}^m \frac{x_k^{n_k}}{[n_k]_q^{s_k}}. \tag{6.2}$$

The substitution $n_k = \sum_{j=k}^m \nu_j$ shows that (6.1) and (6.2) are related by

$$\mathrm{Li}_{s_1,\dots,s_m}[x_1,\dots,x_m] = \lambda_q\begin{bmatrix} s_1,\dots,s_m \\ y_1,\dots,y_m\end{bmatrix}, \qquad y_k = \prod_{j=1}^k x_j^{-1}.$$

Theorem 13 (q-Analog of [3, Theorem 9.1]). *Let $b_1,\dots,b_m \in \mathbf{C}$, $s_1,\dots,s_m > 0$ and let n be a positive integer. Then*

$$n^m \lambda_{q^n}\begin{bmatrix} s_1,\dots,s_m \\ b_1^n,\dots,b_m^n\end{bmatrix} = [n]_q^s \sum_{\varepsilon_1^n=\cdots=\varepsilon_m^n=1} \lambda_q\begin{bmatrix} s_1,\dots,s_m \\ \varepsilon_1 b_1,\dots,\varepsilon_m b_m\end{bmatrix},$$

where the sum is over all n^m sequences $(\varepsilon_1,\dots,\varepsilon_m)$ of complex nth roots of unity, and $s = \sum_{k=1}^m s_k$.

Proof. In light of the identity

$$\frac{1}{[\nu]_{q^n}^s} = \left(\frac{1-q^n}{1-q^{n\nu}}\right)^s = \left(\frac{1-q^n}{1-q}\right)^s \left(\frac{1-q}{1-q^{n\nu}}\right)^s = \frac{[n]_q^s}{[n\nu]_q^s},$$

we have

$$\begin{aligned}
n^m \lambda_{q^n}\begin{bmatrix} s_1,\dots,s_m \\ b_1^n,\dots,b_m^n\end{bmatrix} &= n^m \sum_{\nu_1,\dots,\nu_m>0} \prod_{k=1}^m b_k^{-n\nu_k}\left[\sum_{j=k}^m \nu_j\right]_{q^n}^{-s_j} \\
&= n^m \sum_{\nu_1,\dots,\nu_m>0} \prod_{k=1}^m b_k^{-n\nu_k}[n]_q^{s_j}\left[n\sum_{j=k}^m \nu_j\right]_q^{-s_j} \\
&= [n]_q^s \sum_{\nu_1,\dots,\nu_m>0} \prod_{k=1}^m n b_k^{-n\nu_k}\left[\sum_{j=k}^m n\nu_j\right]_q^{-s_j} \\
&= [n]_q^s \sum_{\nu_1,\dots,\nu_m>0} \prod_{k=1}^m b_k^{-\nu_k}\left[\sum_{j=k}^m \nu_j\right]_q^{-s_j} \sum_{\mu_k=0}^{n-1} e^{-2\pi i\mu_k\nu_k/n} \\
&= [n]_q^s \sum_{\mu_1=0}^{n-1}\cdots\sum_{\mu_m=0}^{n-1} \sum_{\nu_1,\dots,\nu_m>0} \prod_{k=1}^m b_k^{-\nu_k} e^{-2\pi i\mu_k\nu_k/n}\left[\sum_{j=k}^m \nu_j\right]_q^{-s_j}.
\end{aligned}$$

Letting $\varepsilon_k = e^{2\pi i\mu_k/n}$ completes the proof. $\square$

In contrast with our proof of Theorem 13, the proof of the limiting case in [3] made use of the Drinfel'd simplex integral representation for multiple polylogarithms. As integral representations for multiple polylogarithms have proved eminently useful in establishing many of their properties, we derive here a q-analog of the Drinfel'd simplex integral for the multiple q-polylogarithm (6.1). Recall [1, p. 486], [14, p. 19], [22] the Jackson q-integral

$$\int_0^a f(t)\,d_qt := (1-q)\sum_{n=0}^\infty aq^n f\bigl(aq^n\bigr), \quad a>0.$$

Theorem 14. *Let $s_1,\dots,s_m$ be positive integers. For the multiple q-polylogarithm, we have the multiple Jackson q-integral representation*

$$\lambda_q\left[\begin{matrix} s_1,\dots,s_m \\ y_1,\dots,y_m\end{matrix}\right] = \int \prod_{k=1}^m \left(\prod_{r=1}^{s_k-1} \frac{d_q t_r^{(k)}}{t_r^{(k)}}\right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}}, \tag{6.3}$$

where the multiple Jackson q-integral (6.3) *is over the simplex*

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0.$$

Remark 5. As in [3], we may abbreviate (6.3) by

$$\lambda_q\left[\begin{matrix} s_1,\dots,s_m \\ y_1,\dots,y_m\end{matrix}\right] = (-1)^m \int_0^1 \prod_{k=1}^m \bigl(\omega[0]\bigr)^{s_k-1}\omega[y_k], \qquad \omega[b] := \frac{d_qt}{t-b}.$$

Corollary 6. *For multiple q-zeta values, we have the multiple Jackson q-integral representation*

$$\zeta[s_1,\dots,s_m] = (-1)^m \int_0^1 \prod_{k=1}^m \bigl(\omega[0]\bigr)^{s_k-1}\omega\left[\prod_{j=1}^k q^{1-s_j}\right].$$

Proof of Theorem 14. We first establish the following lemma.

Lemma 9. *Let s be a positive integer,* $0<t_0<1$ *and* $m>0$. *Then*

$$\int\limits_{t_0>t_1>\cdots>t_s>0} \left(\prod_{r=1}^{s-1} \frac{d_qt_r}{t_r}\right) t_s^{m-1}\,d_qt_s = \frac{t_0^m}{[m]_q^s}.$$

Proof. When $s=1$, the integral reduces to the geometric series

$$\int\limits_{t_0>t_1>0} t_1^{m-1}\,d_qt_1 = (1-q)t_0\sum_{j=0}^\infty q^j\bigl(q^jt_0\bigr)^{m-1} = \left(\frac{1-q}{1-q^m}\right)t_0^m.$$

Suppose the lemma holds for $s-1$. By the inductive hypothesis,

$$\int\limits_{t_0>t_1>\cdots>t_s>0} \left(\prod_{r=1}^{s-1} \frac{d_q t_r}{t_r}\right) t_s^{m-1}\, d_q t_s = \int\limits_{t_0>t_1>0} \frac{t_1^m}{[m]_q^{s-1}} \frac{d_q t_1}{t_1} = \frac{1}{[m]_q^{s-1}} \int\limits_{t_0>t_1>0} t_1^{m-1}\, d_q t_1$$
$$= \frac{t_0^m}{[m]_q^s},$$

as required. □

To prove (6.3), it will suffice to establish the identity

$$\int \prod_{k=1}^{m} \left(\prod_{r=1}^{s_k-1} \frac{d_q t_r^{(k)}}{t_r^{(k)}}\right) \frac{d_q t_{s_k}^{(k)}}{y_k - t_{s_k}^{(k)}} = \lambda_q \left[\begin{matrix} s_1, \ldots, s_m \\ y_1/t_0, \ldots, y_m/t_0 \end{matrix}\right], \tag{6.4}$$

where the integral (6.4) is over the simplex

$$t_0 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0.$$

When $m=1$, (6.4) reduces to

$$\int\limits_{t_0>t_1>\cdots>t_s>0} \left(\prod_{r=1}^{s-1} \frac{d_q t_r}{t_r}\right) \frac{y^{-1}\, d_q t_s}{1-y^{-1} t_s} = \int\limits_{t_0>t_1>\cdots>t_s>0} \left(\prod_{r=1}^{s-1} \frac{d_q t_r}{t_r}\right) \sum_{\nu=1}^{\infty} y^{-\nu} t_s^{\nu-1}\, d_q t_s$$
$$= \sum_{\nu=1}^{\infty} y^{-\nu} \int\limits_{t_0>\cdots>t_s>0} \left(\prod_{r=1}^{s-1} \frac{d_q t_r}{t_r}\right) t_s^{\nu-1}\, d_q t_s$$
$$= \sum_{\nu=1}^{\infty} \frac{y^{-\nu} t_0^{\nu}}{[\nu]_q^s} = \lambda \left[\begin{matrix} s \\ y/t_0 \end{matrix}\right].$$

Suppose (6.4) holds for $m-1$. Then the inductive hypothesis implies that the integral (6.4) is equal to

$$\int\limits_{t_0>t_1>\cdots>t_{s_1}>0} \left(\prod_{r=1}^{s_1-1} \frac{d_q t_r}{t_r}\right) \frac{y_1^{-1}\, d_q t_{s_1}}{1-y_1^{-1} t_{s_1}} \sum_{\nu_2,\ldots,\nu_m>0} \prod_{k=2}^{m} t_{s_1}^{\nu_k} y_k^{-\nu_k} \left[\sum_{j=k}^{m} \nu_j\right]_q^{-s_j}$$
$$= \sum_{\nu_1,\ldots,\nu_m>0} y_1^{-\nu_1} \prod_{k=2}^{m} y_k^{-\nu_k} \left[\sum_{j=k}^{m} \nu_j\right]_q^{-s_j} \int\limits_{t_0>t_1>\cdots>t_{s_1}>0} \left(\prod_{r=1}^{s_1-1} \frac{d_q t_r}{t_r}\right) t_{s_1}^{\nu_1+\nu_2+\cdots+\nu_m-1}\, d_q t_{s_1}$$

$$= \sum_{\nu_1,\dots,\nu_m>0} \prod_{k=1}^{m} y_k^{-\nu_k} t_0^{\nu_k} \left[\sum_{j=k}^{m} \nu_j \right]_q^{-s_k}$$

$$= \lambda_q \left[\begin{matrix} s_1, \dots, s_m \\ y_1/t_0, \dots, y_m/t_0 \end{matrix} \right]. \qquad \square$$

Remark 6. Zhao [29] has outlined an alternative approach to deriving the multiple Jackson q-integral representation of the multiple q-polylogarithm. In addition, he initiates a study of what are essentially the q-shuffles, first explicated in [6, Section 7], that arise when multiplying two such integrals. Regarding these, the approach taken in [6] is to consider an alphabet A of q-difference forms $f(t)\,d_q t = f(t)(1-q)t$ for various f, and define the q-shuffle product $ш_q$ on the free monoid A^* of words on A by the recursion

$$\begin{cases} \forall w \in A^*, & 1 ш_q w = w ш_q 1 = w, \\ \forall a, b \in A,\ \forall u, v \in A^*, & au ш_q bv = a(u ш_q bv) + b\bigl(\eta(au) ш_q v\bigr). \end{cases}$$

Here, η is the Rogers q-difference operator defined on forms by $\eta(f(t)\,d_q t) = f(qt) \times (1-q)qt$ and extended to an automorphism of A^* in the obvious manner. Using the q-product rule for q-differentiation [14,22] in the form $(D_q fg)(x) = g(x)(D_q f)(x) + f(qx)(D_q g)(x)$, one readily verifies that this definition of the q-shuffle ensures that equation

$$\int_0^x u ш_q v = \left(\int_0^x u \right) \left(\int_0^x v \right),$$

a q-analog of the corresponding shuffle relation for the ordinary Drinfel'd simplex integral, holds for the multiple Jackson q-integral. However, the implications of this definition for multiple q-polylogarithms and multiple q-zeta values have not yet been worked out.

In contrast, Zhao [29] uses the equivalent, but more symmetric form

$$(D_q fg)(x) = f(x)(D_q g)(x) + g(x)(D_q f)(x) + (q-1)x(D_q f)(x)(D_q g)(x)$$

of the q-product rule to derive the formula

$$\left(\int_0^x \prod_{i=1}^{r} a_i \right) \left(\int_0^x \prod_{j=1}^{s} b_j \right)$$
$$= \int_0^x \left(\prod_{i=1}^{r} a_i ш \prod_{j=1}^{s} b_j \right) + \sum_{c=1}^{\min(r,s)} (q-1)^c$$
$$\times \sum_{\substack{1 \leqslant i_1 < \dots < i_c \leqslant r \\ 1 \leqslant j_1 < \dots < j_c \leqslant s}} \int_0^x \prod_{k=1}^{c+1} \bigl\{ \bigl((a_{1+i_{k-1}} \cdots a_{i_k - 1}) ш (b_{1+j_{k-1}} \cdots b_{j_k - 1}) \bigr) \langle a_{i_k}, b_{j_k} \rangle \bigr\},$$

where $ш$ denotes the ordinary shuffle product [3,4,6–8], $a_i = d_q t/(t-\alpha_i)$, $b_j = d_q t/(t-\beta_j)$, $i_0 = j_0 = 0$, $i_{c+1} = r+1$, $j_{c+1} = s+1$, $\langle a_{r+1}, b_{s+1}\rangle = 1$, and for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$,

$$\langle a_i, b_j\rangle = \frac{t\, d_q t}{(t-\alpha_i)(t-\beta_j)} = \begin{cases} \dfrac{1}{\beta_j - \alpha_i}\left(\dfrac{\beta_j\, d_q t}{t-\beta_j} - \dfrac{\alpha_i\, d_q t}{t-\alpha_i}\right), & \text{if } \alpha_i \neq \beta_j, \\ \dfrac{d_q t}{t-\beta} + \dfrac{\beta\, d_q t}{(t-\beta)^2}, & \text{if } \alpha_i = \beta_j = \beta. \end{cases}$$

This is essentially a q-shuffle multiplication rule for the multiple q-polylogarithm, and in principle could lead to a q-shuffle relation for multiple q-zeta values if all terms could be reduced to such. Zhao works out the case of the depth-1 product $\zeta_q(m)\zeta_q(n)$ for $2 \leqslant m, n \in \mathbf{Z}$, but even here the result is quite complicated, and in addition we get non-zeta polylogarithmic terms

$$\sum_{k=1}^{\infty} \frac{q^{(j+1)k}}{[k]_q^2}, \quad 0 < j \in \mathbf{Z},$$

appearing in the final result. Thus, at least for the present, the situation with respect to q-shuffles for multiple q-zeta values is less satisfactory than the corresponding situation in the case of the q-stuffles (Section 2).

7. A double generating function for $\zeta[m+2, \{1\}^n]$

In this section, we derive the following q-analog of [2, Eq. (10)] and a few of its implications.

Theorem 15. *The double generating function identity*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^{m+1} v^{n+1} \zeta\big[m+2, \{1\}^n\big]$$
$$= 1 - \exp\left\{\sum_{k=2}^{\infty}\big\{u^k + v^k - \big(u+v+(1-q)uv\big)^k\big\}\frac{1}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\} \quad (7.1)$$

holds.

Noting that the generating function (7.1) is symmetric in u and v, we immediately derive the following special case of q-duality.

Corollary 7. *For all non-negative integers m and n, $\zeta[m+2, \{1\}^n] = \zeta[n+2, \{1\}^m]$.*

Of course, we have already proved q-duality at full strength (Corollary 3) as a consequence of generalized q-duality (Theorem 5). The main interest for Theorem 15 may be that it shows that $\zeta[m+2,\{1\}^n]$ can be expressed in terms of sums of products of depth-1 q-zeta values. When $n=1$, this reduces to the following convolution identity, which provides a q-analog of Euler's evaluation [2, Eq. (31)], [12,24] of $\zeta(m+2,1)$.

Corollary 8. *Let m be a non-negative integer. Then*

$$2\zeta[m+2,1]=(m+2)\zeta[m+3]+(1-q)\,m\,\zeta[m+2]-\sum_{k=2}^{m+1}\zeta[m+3-k]\,\zeta[k].$$

In particular, when $m=0$ we get $\zeta[2,1]=\zeta[3]$, which corrects an error in [30, Theorem 15].

Proof. Compare coefficients of $u^{m+1}v^2$ on each side of the double generating function identity (7.1). Letting

$$c_k:=\begin{cases}\displaystyle\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j], & \text{if } k\geqslant 2\\ 0, & \text{if } k<2,\end{cases}$$

we find that

$$\begin{aligned}2\zeta[m+2,1]&=(m+2)c_{m+3}+2(1-q)(m+1)c_{m+2}+(1-q)^2\,m\,c_{m+1}\\&\quad-\sum_{k+l=m+3}c_kc_l+2(q-1)\sum_{k+l=m+2}c_kc_l-(q-1)^2\sum_{k+l=m+1}c_kc_l,\end{aligned}\tag{7.2}$$

where convolution sums in (7.2) range over all integers k and l satisfying the indicated relations. Now

$$\begin{aligned}&(m+2)c_{m+3}+2(1-q)(m+1)c_{m+2}+(1-q)^2\,m\,c_{m+1}\\&\quad=(m+2)\sum_{j=2}^{m+3}(q-1)^{m+3-j}\zeta[j]-2(m+1)\sum_{j=2}^{m+2}(q-1)^{m+3-j}\zeta[j]\\&\qquad+m\sum_{j=2}^{m+1}(q-1)^{m+3-j}\zeta[j]\\&\quad=\bigl\{(m+2)-2(m+1)+m\bigr\}\sum_{j=2}^{m+1}(q-1)^{m+3-j}\zeta[j]\\&\qquad+(m+2)\sum_{j=m+2}^{m+3}(q-1)^{m+3-j}\zeta[j]-2(m+1)(q-1)\zeta[m+2]\\&\quad=(m+2)\zeta[m+3]+(1-q)m\zeta[m+2].\end{aligned}$$

In light of (7.2), it now follows that

$$
\begin{aligned}
&2\zeta[m+2,1]-(m+2)\zeta[m+3]-(1-q)m\zeta[m+2]\\
&\quad=-\sum_{k+l=m+3}c_kc_l+2(q-1)\sum_{k+l=m+2}c_kc_l-(q-1)^2\sum_{k+l=m+1}c_kc_l.
\end{aligned}
$$

To avoid having to deal directly with boundary cases, we set $\zeta_+[n]:=\zeta[n]$ $(n\geqslant 2)$ and $(q-1)_+^n=(q-1)^n$ $(n\geqslant 0)$. Then

$$
\begin{aligned}
&2\zeta[m+2,1]-(m+2)\zeta[m+3]-(1-q)m\zeta[m+2]\\
&\quad=-\sum_{k\in\mathbf{Z}}c_{m+3-k}\bigl\{c_k-2(q-1)c_{k-1}+(q-1)^2c_{k-2}\bigr\}\\
&\quad=-\sum_{k\in\mathbf{Z}}c_{m+3-k}\sum_{j\in\mathbf{Z}}\bigl\{(q-1)_+^{k-j}-2(q-1)(q-1)_+^{k-1-j}\\
&\qquad+(q-1)^2(q-1)_+^{k-2-j}\bigr\}\zeta[j]\\
&\quad=-\sum_{k\in\mathbf{Z}}c_{m+3-k}\Biggl\{\zeta_+[k]+\bigl\{(q-1)-2(q-1)\bigr\}\zeta_+[k-1]\\
&\qquad+\sum_{j\leqslant k-2}\bigl\{(q-1)^{k-j}-2(q-1)^{k-j}+(q-1)^{k-j}\bigr\}\zeta_+[j]\Biggr\}\\
&\quad=\sum_{k\in\mathbf{Z}}c_{m+3-k}(q-1)\zeta_+[k-1]-\sum_{k\in\mathbf{Z}}c_{m+3-k}\,\zeta_+[k].
\end{aligned}
$$

We now re-index the latter two sums, replacing k by $m+4-n$ in the first, and k by $m+3-n$ in the second. Thus,

$$
\begin{aligned}
&2\zeta[m+2,1]-(m+2)\zeta[m+3]-(1-q)m\zeta[m+2]\\
&\quad=\sum_{n\in\mathbf{Z}}\zeta_+[m+3-n](q-1)c_{n-1}-\sum_{n\in\mathbf{Z}}\zeta_+[m+3-n]c_n\\
&\quad=\sum_{n\in\mathbf{Z}}\zeta_+[m+3-n]\sum_{j\in\mathbf{Z}}\bigl\{(q-1)(q-1)_+^{n-1-j}-(q-1)_+^{n-j}\bigr\}\zeta_+[j]\\
&\quad=\sum_{n\in\mathbf{Z}}\zeta_+[m+3-n]\Biggl\{\sum_{j\leqslant n-1}\bigl\{(q-1)^{n-j}-(q-1)^{n-j}\bigr\}\zeta_+[j]-(q-1)_+^0\zeta_+[n]\Biggr\}\\
&\quad=-\sum_{n\in\mathbf{Z}}\zeta_+[m+3-n]\zeta_+[n]\\
&\quad=-\sum_{n=2}^{m+1}\zeta[m+3-n]\zeta[n],
\end{aligned}
$$

as claimed. □

Remark 7. Similarly, one could derive an explicit identity for $\zeta[m+2,1,1]$ in terms of depth-1 q-zeta values by comparing coefficients of $u^{m+1}v^3$ in Theorem 15. The resulting identity would be a q-analog of Markett's double convolution identity [23] for $\zeta(m+2,1,1)$.

Alternatively, Corollary 8 can be proved as a simple consequence of the q-stuffle multiplication rule (2.2) and the depth-2 case of the q-sum formula (Corollary 4). Thus,

$$\begin{aligned}\sum_{k=2}^{m+1}\zeta[m+3-k]\zeta[k] &= \sum_{k=2}^{m+1}\bigl\{\zeta[m+3]+(1-q)\zeta[m+2]\\ &\quad+\zeta[m+3-k,k]+\zeta[k,m+3-k]\bigr\}\\ &= m\zeta[m+3]+(1-q)m\zeta[m+2]+2\sum_{\substack{s,t\geqslant 2\\ s+t=m+3}}\zeta[s,t]\\ &= m\zeta[m+3]+(1-q)m\zeta[m+2]\\ &\quad+2\sum_{\substack{s\geqslant 2,\,t\geqslant 1\\ s+t=m+3}}\zeta[s,t]-2\zeta[m+2,1]\\ &= m\zeta[m+3]+(1-q)m\zeta[m+2]+2\zeta[m+3]-2\zeta[m+2,1]\\ &= (m+2)\zeta[m+3]+(1-q)m\zeta[m+2]-2\zeta[m+2,1].\end{aligned}$$

Our proof of Theorem 15 employs techniques from the theory of basic hypergeometric series. For real x and y and non-negative integer n, the asymmetric q-power [22] is given by

$$(x+y)_q^n := \prod_{k=0}^{n-1}\bigl(x+yq^k\bigr), \qquad (x+y)_q^\infty := \lim_{n\to\infty}(x+y)_q^n.$$

The q-gamma function [1, p. 493], [14, p. 16] is defined by

$$\Gamma_q(x)=\frac{(1-q)_q^\infty(1-q)^{1-x}}{(1-q^x)_q^\infty},$$

and the basic hypergeometric function [1, p. 520], [14, p. xv, Eq. (22)] is

$${}_2\phi_1\left[\begin{matrix}q^a,q^b\\ q^c\end{matrix}\,\middle|\,x\right]=\sum_{n=0}^\infty\frac{(1-q^a)_q^n(1-q^b)_q^n}{(1-q^c)_q^n(1-q)_q^n}\,x^n, \quad |x|<1.$$

Heine's q-analog of Gauss's summation formula for the ordinary hypergeometric function [1, p. 522], [14, p. xv, Eq. (23)] may be stated in the form

$$ {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix} \middle| q^{c-a-b}\right] = \frac{\Gamma_q(c)\Gamma_q(c-a-b)}{\Gamma_q(c-a)\Gamma_q(c-b)}, \quad \left|q^{c-a-b}\right| < 1. \tag{7.3}$$

Our first step towards proving Theorem 15 is to establish the following result.

Theorem 16 (q-Analog of [3, Eq. (6.5)]). *Let x and y be real numbers satisfying $|x| < 1$ and $|y| < 1$. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta\left[m+2,\{1\}^n\right] = 1 - \frac{\Gamma_q(1+x)\Gamma_q(1+y)}{\Gamma_q(1+x+y)}. \tag{7.4}$$

Proof. Let L denote the bivariate double generating function on the left-hand side of (7.4). Then

$$\begin{aligned}
L &= -[y]_q \sum_{m=0}^{\infty}(-1)^{m+1}[x]_q^{m+1}\sum_{k=1}^{\infty}\frac{q^{(m+1)k}}{[k]_q^{m+2}}\prod_{j=1}^{k-1}\left(1-\frac{[y]_q}{[j]_q}\right)\\
&= -[y]_q \sum_{m=0}^{\infty}(-1)^{m+1}[x]_q^{m+1}\sum_{k=1}^{\infty}\frac{q^{(m+1)k}}{[k]_q^{m+2}}\prod_{j=1}^{k-1}\frac{[j]_q-[y]_q}{[j]_q}\\
&= -[y]_q \sum_{m=0}^{\infty}(-1)^{m+1}[x]_q^{m+1}\sum_{k=1}^{\infty}\frac{q^{(m+1)k}}{[k]_q^{m+2}}\prod_{j=1}^{k-1}\frac{q^y-q^j}{1-q^j}\\
&= q^y\left(\frac{1-q^{-y}}{1-q}\right)\sum_{m=0}^{\infty}(-1)^{m+1}[x]_q^{m+1}\sum_{k=1}^{\infty}\frac{q^{(m+1)k}}{[k]_q^{m+2}}\cdot\frac{q^{(k-1)y}}{(1-q)_q^{k-1}}\prod_{j=1}^{k-1}\left(1-q^{j-y}\right)\\
&= \sum_{m=0}^{\infty}(-1)^{m+1}[x]_q^{m+1}\sum_{k=1}^{\infty}\frac{q^{(m+1)k}q^{ky}}{[k]_q^{m+1}}\cdot\frac{(1-q^{-y})_q^k}{(1-q)_q^k}.
\end{aligned}$$

Now interchange order of summation, noting that the sum on m is a geometric series. Thus, we find that

$$\begin{aligned}
L &= \sum_{k=1}^{\infty}\frac{q^{ky}(1-q^{-y})_q^k}{(1-q)_q^k}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{(m+1)k}[x]_q^{m+1}}{[k]_q^{m+1}}\\
&= -\sum_{k=1}^{\infty}\frac{q^{ky}(1-q^{-y})_q^k}{(1-q)_q^k}\cdot\frac{q^k[x]_q/[k]_q}{1+q^k[x]_q/[k]_q}\\
&= -\sum_{k=1}^{\infty}\frac{q^{(y+1)k}(1-q^{-y})_q^k}{(1-q)_q^k}\cdot\frac{[x]_q}{[k]_q+q^k[x]_q}
\end{aligned}$$

$$= -\sum_{k=1}^{\infty} \frac{q^{(y+1)k}(1-q^{-y})_q^k}{(1-q)_q^k} \cdot \frac{1-q^x}{1-q^{k+x}}$$

$$= -\sum_{k=1}^{\infty} \frac{q^{(y+1)k}(1-q^{-y})_q^k}{(1-q)_q^k} \cdot \frac{(1-q^x)_q^k}{(1-q^{1+x})_q^k}$$

$$= 1 - {}_2\phi_1\left[\begin{matrix} q^{-y}, q^x \\ q^{1+x} \end{matrix} \middle| q^{1+y}\right].$$

Invoking Heine's formula (7.3) completes the proof. □

To express the right-hand side of (7.4) in the form of an exponentiated power series, we require the following series expansion of the logarithm of the q-gamma function.

Lemma 10. *For real x such that $-1 < x < 1$, we have*

$$\log \Gamma_q(1+x) = -\gamma_q x + \sum_{k=2}^{\infty} \frac{[x]_q^k}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta[j],$$

where

$$\gamma_q := \log(1-q) - \frac{\log q}{1-q} \sum_{n=1}^{\infty} \frac{q^n}{[n]_q}$$

is a q-analog of Euler's constant, γ.

Proof. By definition,

$$\Gamma_q(1+x) = (1-q)^{-x} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{n+x}}.$$

Therefore,

$$\log \Gamma_q(1+x) + x\log(1-q) = -\sum_{n=1}^{\infty} \log\left(\frac{1-q^{n+x}}{1-q^n}\right) = -\sum_{n=1}^{\infty} \log\left(1 + \left(\frac{1-q^x}{1-q^n}\right) q^n\right)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^k [x]_q^k q^{kn}}{k\,[n]_q^k} = \sum_{k=1}^{\infty} \frac{(-1)^k [x]_q^k}{k} \tilde{\zeta}[k], \tag{7.5}$$

where

$$\tilde{\zeta}[k] := \sum_{n=1}^{\infty} \frac{q^{kn}}{[n]_q^k}, \quad k > 0.$$

If we now multiply the identity

$$\frac{q^{jn}}{[n]_q^j} = (q-1)\frac{q^{(j-1)n}}{[n]_q^{j-1}} + \frac{q^{(j-1)n}}{[n]_q^j} \quad \bigl(n, j \in \mathbf{Z}^+\bigr)$$

by $(q-1)^{k-j}$ and sum on n and j, we find that

$$\sum_{j=2}^{k}(q-1)^{k-j}\,\tilde{\zeta}[j] = \sum_{j=2}^{k}(q-1)^{k-j+1}\,\tilde{\zeta}[j-1] + \sum_{j=2}^{k}(q-1)^{k-j}\zeta[j],$$

which telescopes, leaving us with

$$\tilde{\zeta}[k] = (q-1)^{k-1}\,\tilde{\zeta}[1] + \sum_{j=2}^{k}(q-1)^{k-j}\zeta[j], \quad k \geqslant 1. \tag{7.6}$$

If we now substitute (7.6) into (7.5), there comes

$$\begin{aligned}
&\log \Gamma_q(1+x) + x\log(1-q) \\
&\quad = \sum_{k=1}^{\infty}\frac{(-1)^k[x]_q^k}{k}\left\{(q-1)^{k-1}\,\tilde{\zeta}[1] + \sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\} \\
&\quad = -(q-1)^{-1}\,\tilde{\zeta}[1]\log\bigl(1+(q-1)[x]_q\bigr) + \sum_{k=1}^{\infty}\frac{(-1)^k[x]_q^k}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j] \\
&\quad = \frac{x\tilde{\zeta}[1]\log q}{1-q} + \sum_{k=2}^{\infty}\frac{(-1)^k[x]_q^k}{k}\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j].
\end{aligned}$$

In light of the fact that

$$\log\Gamma(1+x) = -\gamma x + \sum_{k=2}^{\infty}\frac{(-1)^k x^k}{k}\zeta(k)$$

and $\lim_{q\to 1-}\Gamma_q(1+x) = \Gamma(1+x)$, it follows that $\lim_{q\to 1-}\gamma_q = \gamma$. Thus, the proof of Lemma 10 is complete. $\square$

Proof of Theorem 15. By Theorem 16 and Lemma 10, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}[x]_q^{m+1}[y]_q^{n+1}\zeta\bigl[m+2,\{1\}^n\bigr] \\
&\quad = 1 - \exp\left\{\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\bigl([x]_q^k + [y]_q^k - [x+y]_q^k\bigr)\sum_{j=2}^{k}(q-1)^{k-j}\zeta[j]\right\}.
\end{aligned}$$

Noting that $[x+y]_q = [x]_q + [y]_q + (q-1)[x]_q[y]_q$, the result now follows on replacing $[x]_q$ by $-u$ and $[y]_q$ by $-v$. $\square$

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