As a follow-up to David Kung’s January 2008 MAA FOCUS article, “What I Learned from…Shutting Up and Listening,” I present here the results of a similar experiment in my senior capstone seminar held this past spring semester. Like Kung, my intention was to have the class struggle with the problem of inscribing a square in a triangle as described and worked out on pages 23-25 of the second edition of How To Solve It: A New Aspect of Mathematical Method, by G. Pólya. I was surprised that none of my students arrived at the usual solutions based on the Intermediate Value Theorem, nor did they duplicate any of the solutions found by the students in Kung’s class. Instead, three quite different solutions emerged.

Erroll’s solution is based on trigonometry. Given a triangle $ABC$, since the sum of the positive angles is $A + B + C = \pi$, it is easy to see that two of the angles must be acute, say $0 < A < \pi/2, 0 < B < \pi/2$. Let $c$ be the length of the side $AB$ opposite angle $C$. We need to exhibit positive $r, s, t$ satisfying $c = r + s + t$ and such that a square erected on the base $s$ is in fact an inscribed square.

To do this, let $\alpha = \tan A = s/r$ and let $\beta = \tan B = s/t$. Then $r = cr/(r + s + t) = (cs/\alpha)/(s/\alpha + s + s/\beta) = c\beta/(\beta + \alpha \beta + \alpha), t = ct/(r + s + t) = (cs/\beta)/(s/\alpha + s + s/\beta) = c\alpha/(\beta + \alpha \beta + \alpha), s = c - r - t = c - c\beta/(\beta + \alpha \beta + \alpha) - c\alpha/(\beta + \alpha \beta + \alpha) = c\alpha \beta/(\beta + \alpha \beta + \alpha)$. Since $r, s, t$ evidently satisfy the required conditions and are expressed in terms of only the tangents of the given angles $A$ and $B$, the problem is solved.

Dale’s solution is based on similar triangles. It recalls one of the solutions based on the Intermediate Value Theorem, but rather than inferring the existence of the inscribed square, an explicit construction is provided.

Given triangle $ABC$, we again suppose that angles $A$ and $B$ are acute. Choose points $D$ and $G$ on sides $AB$ and $AC$, respectively, so that the square $DEFG$ has base $DE$ on $AB$ and the point $F$ is an interior point of the triangle. Join points $A$ and $F$ with a straight line segment, extending the segment to meet $CB$ at $J$. Drop a perpendicular from $J$ to meet $AB$ at $I$. Draw a line segment perpendicular to $IJ$ through $J$, meeting $AC$ at $K$. Drop a perpendicular from $K$, meeting $AB$ at $H$. We claim that $HIJK$ is a square.

To see this, observe that triangles $AIJ$ and $AEF$ are similar. This implies that $IJ/EF = AJ/AF$. Since triangles $AJK$ and $AFG$ are also similar, we also have that $JK/FG = AJ/AF$. It follows that $IJ/EF = JK/FG$.
Cross-multiplying shows that $\frac{IJ}{JK} = \frac{EF}{FG}$, which is equal to 1 since $DEFG$ is a square. It follows that $HIJK$ is also a square.

Sarah’s solution employed coordinate geometry.

Given a triangle $ABC$, assume again that angles $A$ and $B$ are acute. Without loss of generality, we may suppose that the Cartesian coordinates of the points $A$, $B$, and $C$ are given by $(0, 0)$, $(b, 0)$, and $(a, c)$, respectively, where $0 < a < b$ and $c > 0$. Now choose $t$ satisfying $0 < t < a$ and locate the point $D$ on $AB$ so that $D$ has coordinates $(t, 0)$. Next, locate the point $G$ on $AC$ so that $DG$ is perpendicular to $AD$. Since the line extending $AC$ has equation $y = \frac{cx}{a}$, $G$ has coordinates $(t, \frac{ct}{a})$. Since we want a square with one side $DG$, the next step is to locate the point $E$ on $AB$ so that $DE = DG$. This implies that $E$ has coordinates $(t + \frac{ct}{a}, 0)$. The fourth vertex, $F$, of our square is located at the intersection of the horizontal line through $G$ and the vertical line through $E$. Therefore, $F$ has coordinates $(t + \frac{ct}{a}, \frac{ct}{a})$.

It remains to show that there exists a value of $t$ such that $F$ lies on $CB$. Since the line extending $CB$ has equation $y = \frac{(x - b)c}{(a - b)}$, we need to ensure that the coordinates of $F$ satisfy this equation. It suffices to show that the equation $(t + \frac{ct}{a} - b)\frac{c}{(a - b)} = \frac{ct}{a}$ has a solution $t$ that satisfies the original stipulation $0 < t < a$. Middle-school algebra reveals that the equation has the unique solution $t = ab/(b + c)$. Since $0 < b/(b + c) < 1$, we see that $0 < t < a$, as required.

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*May 20 - June 1, 2009*

Ancient Egypt will surround you with all of its fascination and intrigue while we experience the splendors of Giza which includes the Great Pyramids and mysterious Sphinx and learn of the mathematical genius that was involved in building such structures. A lecture by Dr. Zahi Hawass, Secretary General of the Supreme Council of Antiquities and Director of the Giza Pyramid Excavations, will begin this incredible journey.

We will gain insight into Cairo’s religious past, including visits to the oldest Christian church in Egypt, the Ibn Ezra Synagogue, and the Alabaster Mosque. A visit to the Egyptian Museum of Antiquities will bring us face to face with the mask from Tutankhamun’s gold and precious stone encrusted mummy that no longer is allowed to exhibit outside of Egypt.

The magnificent temples of Karnak are an impressive example of how mathematics was used to lay out the colonnades which provides the rising sunbeams the appropriate angles for the first rays to illuminate the holy places of the temple. Navigating the Nile River will bring us to Edfu and Kom Ombo where the carvings on the temples of Sobeck and Haroeris are believed to be one of the first representations of a lunar calendar. Our tour will include Alexandria, where Hypatia, the first woman to make significant contributions to mathematics, headed the Platonist School and assisted her father, Theon of Alexandria, the mathematician and philosopher.

Jim Ritter, our expert throughout the tour and professor at the Université of Paris, will show us the different ancient Egyptian systems of measure that are evident at Ny-ankh -Knum and Khnum-hotep (the two brother’s tombs) in Sakkara.

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