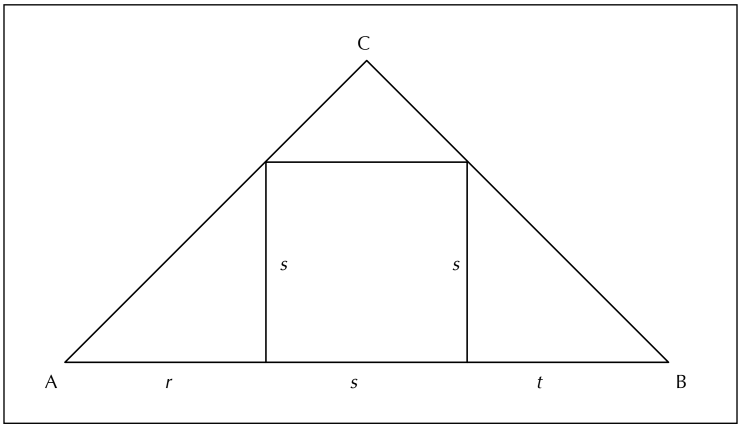
On Shutting Up and Listening

David M. Bradley

As a follow-up to David Kung's January 2008 *MAA FOCUS* article, "What I Learned from…Shutting Up and Listening," I present here the results of a similar experiment in my senior capstone seminar held this past spring semester. Like Kung, my intention was to have the class struggle with the problem of inscribing a square in a triangle as described and worked out on pages 23-25 of the second edition of *How To Solve It: A New Aspect of Mathematical Method*, by G. Pólya. I was surprised that none of my students arrived at the usual solutions based on the Intermediate Value Theorem, nor did they duplicate any of the solutions found by the students in Kung's class. Instead, three quite different solutions emerged.



Erroll's solution is based on trigonometry. Given a triangle *ABC*, since the sum of the positive angles is $A + B + C = \pi$, it is easy to see that two of the angles must be acute, say $0 < A < \pi/2$, $0 < B < \pi/2$. Let *c* be the length of the side *AB* opposite angle *C*. We need to exhibit positive *r*, *s*, and *t* satisfying $c = r + s + t$ and such that a square erected on the base *s* is in fact an inscribed square.

To do this, let $\alpha = \tan A = s/r$ and let $\beta = \tan B = s/t$. Then $r = cr/(r + s + t) = (cs/\alpha)/(s/\alpha + s + s/\beta) = c\beta/(\beta + \alpha\beta + \alpha)$, $t = ct/(r + s + t) = (cs/\beta)/(s/\alpha + s + s/\beta) = c\alpha/(\beta + \alpha\beta + \alpha)$, $s = c - r - t = c - c\beta/(\beta + \alpha\beta + \alpha) - c\alpha/(\beta + \alpha\beta + \alpha) = c\alpha\beta/(\beta + \alpha\beta + \alpha)$. Since *r*, *s*, *t* evidently satisfy the required conditions and are expressed in terms of only the tangents of the given angles *A* and *B*, the problem is solved.

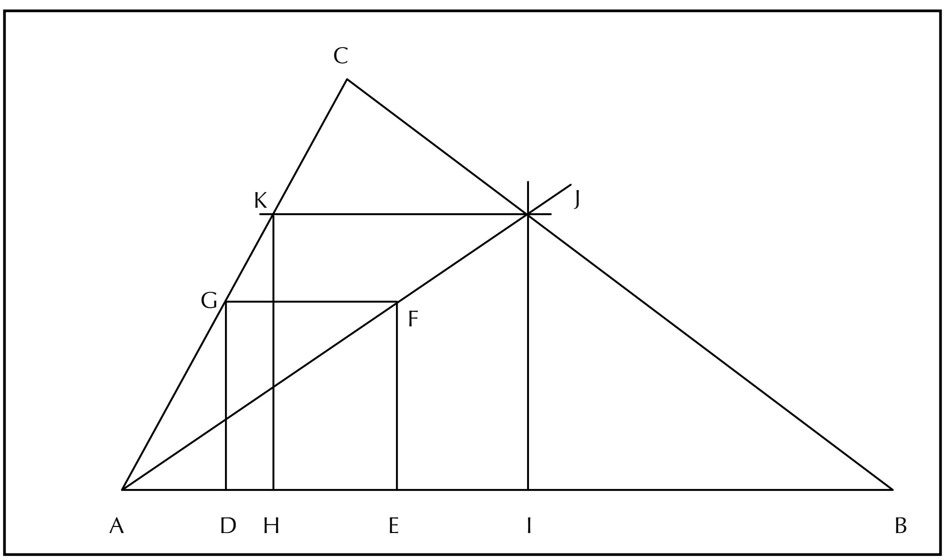
A Problem of Construction

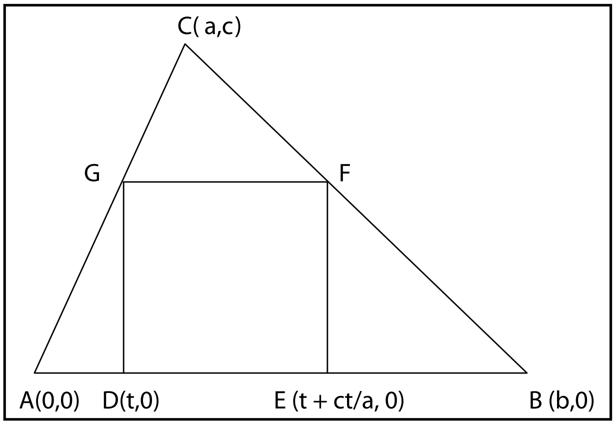
Inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle, the other two vertices of the square on the other two sides of the triangle, one on each.

Dale's solution is based on similar triangles. It recalls one of the solutions based on the Intermediate Value Theorem, but rather than inferring the existence of the inscribed square, an explicit construction is provided.

Given triangle *ABC*, we again suppose that angles *A* and *B* are acute. Choose points *D* and *G* on sides *AB* and *AC*, respectively, so that the square *DEFG* has base *DE* on *AB* and the point *F* is an interior point of the triangle. Join points *A* and *F* with a straight line segment, extending the segment to meet *CB* at *J*. Drop a perpendicular from *J* to meet *AB* at *I*. Draw a line segment perpendicular to *IJ* through *J*, meeting *AC* at *K*. Drop a perpendicular from *K*, meeting *AB* at *H*. We claim that *HIJK* is a square.

To see this, observe that triangles *AIJ* and *AEF* are similar. This implies that $IJ/EF = AJ/AF$. Since triangles *AJK* and *AFG* are also similar, we also have that $JK/FG = AJ/AF$. It follows that $IJ/EF = JK/FG$.





Cross-multiplying shows that $IJ/JK = EF/FG$, which is equal to 1 since $DEFG$ is a square. It follows that $HIJK$ is also a square.

Sarah's solution employed coordinate geometry. Given a triangle ABC, assume again that angles A and B are acute. Without loss of generality, we may suppose that the Cartesian coordinates of the points A, B, and C are given by $(0, 0)$, $(b, 0)$, and (a, c), respectively, where $0 < a < b$ and $c > 0$. Now choose t satisfying $0 < t < a$ and locate the point D on AB so that D has coordinates $(t, 0)$. Next, locate the point G on AC so that DG is perpendicular to AD. Since the line extending AC has equation $y = cx/a$, G has coordinates $(t, ct/a)$. Since we want a square with one side DG, the next step is to locate the point E on AB so that $DE = DG$. This implies that E has coordinates $(t + ct/a, 0)$. The fourth vertex, F, of our square is located at the intersection of the horizontal line through G and the vertical line through E. Therefore, F has coordinates $(t + ct/a, ct/a)$.

It remains to show that there exists a value of t such that F lies on CB. Since the line extending CB has equation $y = (x - b)c/(a - b)$, we need to ensure that the coordinates of F satisfy this equation. It suffices to show that the equation $(t + ct/a - b)c/(a - b) = ct/a$ has a solution t that satisfies the original stipulation $0 < t < a$. Middle-school algebra reveals that the equation has the unique solution $t = ab/(b + c)$. Since $0 < b/(b + c) < 1$, we see that $0 < t < a$, as required.

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