A Difference Differential Equation of Euler-Cauchy Type

David M. Bradley† *and Harold G. Diamond*

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Abstract. We study a class of advanced argument linear difference differential equations analogous to Euler-Cauchy ordinary differential equations. Solutions of two equations of this type have arisen as adjoint functions in sieve theory, and they are also of use in control theory. Here we study the problem in a general setting. Subject to mild assumptions, each of our equations is shown to have a unique solution which is analytic in the right half-plane. In some cases the solution is a polynomial, and in others it has an asymptotic expansion. Finally, the solution is shown to have a representation as an exponential of a Hellinger type integro-differential operator acting on a monomial.

0. Introduction

We are going to study a class of linear difference differential equations with multiple advanced arguments. These equations are analogous to Euler-Cauchy ordinary differential equations. Two examples are

$$(uq(u))' = \kappa q(u) + \kappa q(u+1) \tag{0.1}$$

and

$$(up(u))' = \kappa p(u) - \kappa p(u+1), \tag{0.2}$$

which occur in articles on sieves by Iwaniec [12], Diamond-Halberstam-Richert [4–11], and others [2, 3, 13, 15, 16, 17]. Here, u and κ are real and positive. (The role of κ in sieve theory is to measure the average number of residue classes being deleted for each prime used in the sifting.) Here we consider a multi-parameter generalization which encompasses both equations (0.1) and (0.2). We establish uniqueness results in quite general classes and give two representations of a solution. One of these is a generalization of the integral solutions

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Iwaniec gave for (0.1) and (0.2). The other is in the form of an exponential of a Hellinger type integro-differential operator acting on a monomial.

Throughout, m is a fixed non-negative integer, $0 = v_0 < v_1 < \cdots < v_m$ are fixed real numbers, $\alpha_0, \alpha_1, \ldots, \alpha_m$ are fixed complex constants, and $\beta := \alpha_0 + \alpha_1 + \cdots + \alpha_m$. Consider the difference differential equation

$$uq'(u) = \sum_{j=0}^{m} \alpha_j q(u + v_j), \qquad u > 0. \tag{0.3}$$

Both (0.1) and (0.2) are special cases with $m = 1$, $v_1 = 1$, $\alpha_0 = \kappa - 1$, $\alpha_1 = \pm\kappa$. However, the character of the solutions to (0.1) and (0.2) is radically different, due to the nature of the parameter β. In (0.1), $\beta = 2\kappa - 1$, whereas in (0.2), $\beta = -1$. In general, the solution we exhibit for (0.3) has markedly different qualitative behaviour depending on the sign of $\Re(\beta)$, the analysis being greatly simplified in the case $\Re(\beta) < 0$. The case when β is a non-negative integer is an exception, for then the solution reduces to a polynomial of degree β, which we study in §3. In §4 and §5, we prove some uniqueness theorems, culminating in a result that says (0.3) has essentially only one polynomially bounded solution. In §6, the behaviour of q at 0 is analyzed, an asymptotic expansion at infinity is derived, and a few properties of the coefficients are proved. In §7, we give a representation of q in terms of a Hellinger type integro-differential operator.

1. Background Facts and Additional Notation

We make extensive use of Hankel's formula

$$u^{\beta} = \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\!\circlearrowleft} z^{-\beta-1} e^{uz}\, dz, \qquad u > 0, \qquad \beta \in \mathbf{C} \setminus \{-1, -2, \ldots\}. \tag{1.1}$$

The contour in (1.1), and in the sequel, starts at $-\infty$, hugs the lower side of the negative real axis, then circles the origin in the positive (counter-clockwise) direction before returning to $-\infty$ along the upper side of the negative real axis. If β is a negative integer, (1.1) should be regarded as a limit, the form of which can be seen by specializing β in the following formula which is valid when $\Re(\beta) < 0$:

$$u^{\beta} = \frac{1}{\Gamma(-\beta)} \int_0^{\infty} x^{-\beta-1} e^{-ux}\, dx, \qquad u > 0.$$

We also require the following relationship [1, 14] between exponential integrals:

$$\int_0^{w} \frac{1 - e^{-t}}{t}\, dt = \log w + \gamma + \mathrm{E}_1(w), \qquad |\arg w| < \pi, \tag{1.2}$$

where

$$\mathrm{E}_1(w) := \int_w^\infty \frac{e^{-t}}{t}\,dt \tag{1.3}$$

and $\gamma = 0.57721566\ldots$ denotes Euler's constant. Note that the left-hand side of (1.2) is entire.

Finally, where convenient we denote the coefficient of $z^n/n!$ in the power series $F(z)$ by $[z^n/n!]F(z)$. Clearly, if F is analytic in a neighbourhood of the origin, then $[z^n/n!]$, $n![z^n]$, and $(\partial/\partial z)^n|_{z=0}$ are equivalent operators.

2. The Function $q(u)$

First, observe that the difference differential equation (0.3) is homogeneous. That is, if q is a solution, then so is Cq, for any complex constant C. Next, when $m = 0$ and $\alpha_0 = \alpha$, (0.3) reduces to

$$uq'(u) = \alpha q(u)$$

with general solution $q(u) = Cu^\alpha$. Suppose now we seek a solution to (0.3) of the form $q(u) = Cu^b$ for some constants C and b with $C \neq 0$. Then (0.3) gives

$$bu^b = \sum_{j=0}^m \alpha_j (u+v_j)^b.$$

If this latter "equation" is to hold even approximately, then letting $u \to \infty$, we must have $b = \sum_{j=0}^m \alpha_j = \beta$. The previous heuristic remarks are made rigorous in the following

Proposition 2.1. *Let b and C be complex constants with $C \neq 0$. Suppose (0.3) has a solution which satisfies*

$$q(u) \sim Cu^b, \qquad u \to \infty \tag{2.0}$$

Then $b = \beta = \sum_{j=0}^m \alpha_j$.

Proof. We may as well assume $C = 1$. Since it is generally not permissible to differentiate an asymptotic relation, let $c > |b|$ and write (0.3) in the form

$$(u^c q(u))' = u^{c-1}\left(cq(u) + \sum_{j=0}^m \alpha_j q(u+v_j)\right) \sim u^{b+c-1}\left(c + \sum_{j=0}^m \alpha_j\right).$$

Integrating,

$$u^{b+c} \sim u^c q(u) - q(1) \sim \frac{u^{b+c}-1}{b+c}\left(c + \sum_{j=0}^m \alpha_j\right).$$

It follows that $b = \sum_{j=0}^{m} \alpha_j$, as claimed.

We now exhibit a solution to (0.3) having the asymptotic property in Proposition 2.1.

Proposition 2.2. *Let $\beta = \sum_{j=0}^{m} \alpha_j$. Then (0.3) has a solution satisfying $q(u) \sim u^\beta$ for $u \to \infty$ given by*

$$q(u) = \frac{\Gamma(\beta+1)}{2\pi i} \int_{\Longrightarrow\!\circlearrowleft} z^{-\beta-1} \exp\left\{ uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t} - 1}{t}\, dt \right\} dz, \qquad u > 0. \tag{2.1}$$

Remarks. Since $v_0 = 0$, it makes no difference whether the sum in (2.1) starts at $j = 0$ or $j = 1$. The argument of the exponential is independent of α_0. As in (1.1), the integral should be understood as a limit if β is a negative integer. For now, this limit will not concern us, as it is easily determined from Corollary 2.4 below. Since the singularities at negative integer β values are removable, q as given by (2.1) is an entire function of each α_j. Also, note that the right side of (2.1) defines an analytic function of u in the half-plane $\Re(u) > 0$. Furthermore, Proposition 2.3 below shows that if each α_j is real and $u > 0$, then $q(u)$, as we might expect from (0.3), is real.

Proof. One can, of course, verify the solution (2.1) by differentiating under the integral. This operation is justified by uniform convergence for u in a compact subset of $]0, \infty[$. That the asymptotic requirement is fulfilled follows from Hankel's formula (1.1) and the fact that the integrand behaves like $z^{-\beta-1}e^{uz}$ for small $|z|$ and large u.

A more instructive proof is the following, where we indicate how one might actually discover the solution (2.1). In view of Hankel's formula (1.1) and the normalizing requirement, one is motivated to *guess* a solution of the form

$$q(u) = \frac{\Gamma(\beta+1)}{2\pi i} \int_{\Longrightarrow\!\circlearrowleft} z^{-\beta-1} e^{uz} f(z)\, dz, \qquad u > 0, \tag{2.2}$$

where f is some function to be determined. Let us assume that f is continuously differentiable and for all $\varepsilon > 0$,

$$|f(z)| \ll e^{\varepsilon |z|} \tag{2.3}$$

on the integration contour. Then differentiating under the integral is permissible, and yields

$$q'(u) = \frac{\Gamma(\beta+1)}{2\pi i} \int_{\Longrightarrow\!\circlearrowleft} z^{-\beta} e^{uz} f(z)\, dz.$$

A subsequent integration by parts produces

$$uq'(u) = \frac{\Gamma(\beta+1)}{2\pi i} \int_{\Longrightarrow\!\circlearrowleft} z^{-\beta-1} e^{uz} \left\{ \beta f(z) - z f'(z) \right\} dz.$$

The integrated term vanishes due to its exponential rate of convergence on the contour. On the other hand, from (0.3) and (2.2), we have

$$uq'(u) = \frac{\Gamma(\beta+1)}{2\pi i} \int z^{-\beta-1} e^{uz} f(z) \sum_{j=0}^{m} \alpha_j e^{v_j z}\, dz.$$

Subtracting the last two expressions for $uq'(u)$ yields

$$\frac{\Gamma(\beta+1)}{2\pi i} \int z^{-\beta-1} e^{uz} \Big\{ f(z) \sum_{j=0}^{m} \alpha_j e^{v_j z} - \beta f(z) + z f'(z) \Big\}\, dz = 0.$$

We use the preceding equation to define f. We set the braced expression equal to zero and solve the resulting separable first-order ordinary differential equation for f. We get,

$$f(z) = f(0) \exp\Big\{ -\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}, \tag{2.4}$$

and we obtain a solution to (0.3) given by

$$q(u) = f(0) \frac{\Gamma(\beta+1)}{2\pi i} \int z^{-\beta-1} \exp\Big\{ uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}\, dz.$$

Note that f as given by (2.4) is continuously differentiable (in fact, f is entire) and satisfies the hypothesis (2.3). We can determine the constant $f(0)$ by letting $u \to \infty$ and using the normalizing condition $q(u) \sim u^\beta$. As $u \to \infty$, the main contribution to the integral occurs in a small neighbourhood of the origin, where

$$\int_0^z \frac{e^{v_j t}-1}{t}\, dt = v_j z + O(z^2), \qquad |z| \to 0.$$

Thus,

$$f(z) = f(0)\,(1 + O(z)), \qquad |z| \to 0,$$

and so by Hankel's formula (1.1), as $u \to \infty$, we have

$$q(u) = f(0) \frac{\Gamma(\beta+1)}{2\pi i} \int z^{-\beta-1} e^{uz} \,(1 + O(z))\, dz = f(0) u^\beta \,(1 + O(1/u)),$$

from which it follows that $f(0) = 1$.

Proposition 2.3. *Let n be a non-negative integer such that $n > \Re(\beta)$. Then the loop integral representation (2.1) of Proposition 2.2 can be replaced by*

$$q(u) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_0^{\infty} x^{n-\beta-1} \left(\frac{\partial}{\partial x}\right)^n \exp\left\{ -ux + \sum_{j=0}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \right\} dx. \tag{2.5}$$

Proof. Integrate (2.1) by parts n times. The integrated terms all vanish due to the presence of e^{uz}, which occurs as a factor of every derivative of

$$\exp\left\{ uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \right\}.$$

It follows that

$$q(u) = \frac{(-1)^n}{(-\beta)(-\beta+1)\cdots(-\beta+n-1)} \times \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{n-\beta-1} \left(\frac{\partial}{\partial z}\right)^n \exp\left\{ uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \right\} dz,$$

which should be understood as a limit if β is an integer. Since $n > \Re(\beta)$, we may collapse the contour onto the negative real axis. After making the change of variable $x = -z$, there results

$$q(u) = \frac{(-1)^n \sin(\pi(\beta+1))}{(-\beta)(-\beta+1)\cdots(-\beta+n-1)} \times \frac{\Gamma(\beta+1)}{2\pi i} \int_0^{\infty} x^{n-\beta-1} \left(\frac{\partial}{\partial x}\right)^n \exp\left\{ -ux + \sum_{j=0}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \right\} dx.$$

Now write the product $(-\beta)(-\beta+1)\cdots(-\beta+n-1)$ in the form $\Gamma(n-\beta)/\Gamma(-\beta)$ and apply the reflection formula for the gamma function to complete the proof.

Corollary 2.4. *If $\Re(\beta) < 0$, then (2.1) has the following Laplace transform representation:*

$$q(u) = \frac{1}{\Gamma(-\beta)} \int_0^{\infty} e^{-ux} x^{-\beta-1} \exp\left\{ \sum_{j=0}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \right\} dx, \qquad u > 0. \tag{2.6}$$

Corollary 2.5. *If each α_j is real and $\beta < 0$, then q is positive, convex, and log-convex.*

Proof. The conditions on the α_j and β ensure that the Laplace transform representation (2.6) is valid, and that the integrand is real and non-negative. By Hölder's inequality, q, being the Laplace transform of a non-negative function, is log-convex. Now log-convexity of q implies that $qq'' \geqslant (q')^2$, so in particular, q and q'' have the same sign. Since obviously (2.6) implies $q(u) \geqslant 0$, it follows that $q''(u) \geqslant 0$ i.e. q is convex.

Alternatively, differentiating (2.6) twice under the integral with respect to u is valid and shows directly that $q''(u) \geqslant 0$ for $u > 0$. Log-convexity then follows on applying the Schwarz inequality to the Laplace integral representation of q'.

3. The Polynomials $Q_n(u)$

We return to the contour integral representation (2.1). If β is a non-negative integer, then the only singularity of the integrand, a branch point at $z = 0$, is a pole, and we can close the contour and deform it into the unit circle, obtaining

$$
\begin{aligned}
q(u) &= \frac{\Gamma(\beta+1)}{2\pi i} \oint_{|z|=1} z^{-\beta-1} \exp\left\{uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} dz \\
&= \left(\frac{\partial}{\partial z}\right)^{\beta}\Bigg|_{z=0} \exp\left\{uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}, \qquad (3.1)
\end{aligned}
$$

a polynomial of degree β. Motivated by the preceding treatment of q for special values of β, for general β we define polynomials $Q_n(u)$ for $n = 0, 1, 2, \ldots$ by

$$
\begin{aligned}
Q_n(u) &:= \left(\frac{\partial}{\partial z}\right)^{n}\Bigg|_{z=0} \exp\left\{uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\
&= \left[\frac{z^n}{n!}\right] \exp\left\{uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}, \qquad (3.2)
\end{aligned}
$$

(cf. Wheeler [16] for the $m = 1$ case). As in (2.1), it makes no difference whether the sum in (3.2) starts at $j = 0$ or $j = 1$. The polynomials Q_n are independent of α_0.

Proposition 3.1. *For all non-negative integers n, and all complex numbers u, v, we have*

$$
Q_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} Q_k(u) v^{n-k}.
$$

In particular, taking $u = 0$ gives a formula for $Q_n(v)$ in terms of the constants $Q_n(0)$.

Proof. We have

$$
\begin{aligned}
Q_n(u+v) &= \left(\frac{\partial}{\partial z}\right)^{n}\Bigg|_{z=0} \exp\left\{(u+v)z - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\
&= \left(\frac{\partial}{\partial z}\right)^{n}\Bigg|_{z=0} \exp\left\{uz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \exp(vz).
\end{aligned}
$$

The result now follows from Leibniz's rule.

Next, we'd like to develop a convenient recurrence formula for the polynomials Q_n.

Theorem 1. *For all positive integers n and complex u, we have*

$$Q_n(u) = n\int_0^u Q_{n-1}(t)\,dt - \sum_{j=0}^m \alpha_j \int_0^{v_j} Q_{n-1}(t)\,dt. \tag{3.3}$$

Remark. Again, since $v_0 = 0$, it makes no difference whether the sum in (3.3) starts at $j = 0$ or at $j = 1$.

Proof. Note that for $1 \leqslant n \in \mathbf{Z}$,

$$\begin{aligned} Q_n'(u) &= \left[\frac{z^n}{n!}\right] z \exp\left\{uz - \sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\ &= n\left[\frac{z^{n-1}}{(n-1)!}\right] \exp\left\{uz - \sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\ &= nQ_{n-1}(u). \end{aligned}$$

Integrating, it follows that

$$Q_n(u) = Q_n(0) + n\int_0^u Q_{n-1}(t)\,dt. \tag{3.4}$$

We need to express $Q_n(0)$ in terms of Q_{n-1}. To this end, consider

$$\begin{aligned} &\frac{d}{dz}\exp\left\{-\sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} = \\ &\qquad \exp\left\{-\sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \sum_{j=0}^m \alpha_j \left(\frac{1-e^{v_j z}}{z}\right). \end{aligned}$$

Thus for $1 \leqslant n \in \mathbf{Z}$,

$$\begin{aligned} nQ_n(0) &= n\left[\frac{z^n}{n!}\right] \exp\left\{-\sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\ &= n\left[\frac{z^{n-1}}{(n-1)!}\right] \frac{d}{dz}\exp\left\{-\sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\ &= n\left[\frac{z^{n-1}}{(n-1)!}\right] \exp\left\{-\sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \sum_{j=0}^m \alpha_j \left(\frac{1-e^{v_j z}}{z}\right) \end{aligned}$$

$$= \left[\frac{z^n}{n!}\right] \exp\left\{-\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \sum_{j=0}^{m} \alpha_j \left(1-e^{v_j z}\right)$$

$$= \left[\frac{z^n}{n!}\right] \exp\left\{-\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \left(\beta - \sum_{j=0}^{m} \alpha_j e^{v_j z}\right)$$

$$= \beta Q_n(0) - \sum_{j=0}^{m} \alpha_j Q_n(v_j). \tag{3.5}$$

In particular, we have the interesting result that if $\beta = n$, then

$$\sum_{j=0}^{m} \alpha_j Q_n(v_j) = 0. \tag{3.6}$$

Recall our aim is to replace $Q_n(0)$ in (3.4) with an expression involving Q_{n-1}. Using (3.4) in (3.5), we have

$$\begin{aligned}(\beta - n) Q_n(0) &= \sum_{j=0}^{m} \alpha_j Q_n(v_j) \\ &= \sum_{j=0}^{m} \alpha_j Q_n(0) + n \sum_{j=0}^{m} \alpha_j \int_0^{v_j} Q_{n-1}(t)\,dt\end{aligned}$$

Using $\beta = \sum_{j=0}^{m} \alpha_j$, and $n \geqslant 1$, we can solve the above expression for $Q_n(0)$. Thus,

$$Q_n(0) = -\sum_{j=0}^{m} \alpha_j \int_0^{v_j} Q_{n-1}(t)\,dt.$$

We now substitute this latter expression in (3.4) and the proof is complete.

Using the recurrence, we can easily generate

$$\begin{aligned}&Q_0(u) = 1, \\ &Q_1(u) = u - \sum_{j=0}^{m} \alpha_j v_j, \\ &Q_2(u) = u^2 - 2u \sum_{j=0}^{m} \alpha_j v_j - \frac{1}{2} \sum_{j=0}^{m} \alpha_j v_j^2 + \left(\sum_{j=0}^{m} \alpha_j v_j\right)^2,\end{aligned}$$

etc.

Theorem 2. *Let $\beta = n$ be a non-negative integer. Then up to multiplication by an arbitrary constant factor, the polynomial Q_n given by (3.2) (equivalently, the polynomial q given by (3.1)) is the unique polynomial solution to the difference differential equation (0.3). In* *other words, if p is a polynomial which satisfies (0.3), then p is a constant multiple of the polynomial q of (3.1).*

Proof. Suppose that p_1 and p_2 are non-zero polynomial solutions of (0.3). Then for $i = 1, 2$,

$$p_i(u) \sim C_i u^{d_i}, \qquad u \to \infty$$

where d_i is the degree of p_i and $C_1 \neq 0 \neq C_2$. By Proposition 2.1, $d_1 = d_2 = \beta$. Now let $p := C_1^{-1}p_1 - C_2^{-1}p_2$. Then p is also a solution to (0.3) with degree strictly less than β. By Proposition 2.1 again, it follows that p is the zero polynomial, i.e. $p_1 = C_1 C_2^{-1} p_2$. Thus we have shown that any two polynomial solutions of (0.3) differ by at most an arbitrary constant factor. Since we've already seen that Q_n (and hence CQ_n for any constant C) is a solution to (0.3) with $\beta = n$, the proof is complete.

We prove some broader uniqueness theorems in the following sections.

4. A Uniqueness Result

In [8], the difference differential equations (0.1) and (0.2) with $\kappa \geqslant 1$ were shown to have unique solutions subject to the normalizing condition (2.0) with $C = 1$. Here, we prove a corresponding result for (0.3) but with no corresponding restriction on the parameters.

Theorem 3. *There exists at most one function q which satisfies the difference differential equation (0.3) and the normalizing condition (2.0) with $C = 1$.*

Proof. Let $c := \Re(\beta)$, $M := (m+1)\max_j |\alpha_j|$, $v := v_m = \max_j v_j$, $\alpha := \alpha_0$. It will be convenient to assume $m \geqslant 1$ and some $\alpha_j \neq 0$ so that M and v are both positive. Otherwise, (0.3) reduces to the ordinary differential equation $uq'(u) = \alpha q(u)$ whose behaviour is well-known. Write (0.3) in the form

$$\big(u^{-\beta}q(u)\big)' = u^{-\beta-1}\sum_{j=0}^{m} \alpha_j\{q(u+v_j) - q(u)\}, \qquad u > 0. \tag{4.1}$$

Now suppose that q_1 and q_2 satisfy both (0.3) and (2.0) with $C = 1$. By Proposition 2.1, both functions satisfy (2.0) with $b = \beta$. Then $q := q_1 - q_2$ satisfies (0.3) and $q(u) = o(u^c)$ as $u \to \infty$. We need to show that $q = 0$. Let $B > 0$ satisfy

$$|q(u)| < Bu^c, \qquad u > u_0 \tag{4.2}$$

where u_0 does not depend on B, and will be chosen later. We show by an iterative argument that B can be made arbitrarily small. Integrating (4.1), we obtain

$$-u^{-\beta}q(u) = \sum_{j=0}^{m} \alpha_j \int_u^\infty t^{-\beta-1}\{q(t+v_j) - q(t)\}\,dt,$$

which implies that

$$|q(u)| \leqslant Mu^c \int_u^\infty t^{-c-1} |vq'(t+\theta)|\, dt, \tag{4.3}$$

where $\theta := \theta(t) \in [0, v]$. We need to estimate q' in the integral. We distinguish three cases: $c \geqslant 1$, $0 \leqslant c < 1$, and $c < 0$.

Case $c \geqslant 1$**.** From the difference differential equation (0.3), we obtain

$$(u-v)|q'(u)| \leqslant u|q'(u)| \leqslant \sum_{j=0}^m |\alpha_j||q(u+v_j)| \leqslant MB(u+v)^c.$$

Now

$$\left(\frac{u+v}{u-v}\right)^c < 2 \quad \text{if} \quad u > v\left(1 + \frac{2c}{\log 2}\right) \tag{4.4}$$

and so we find that

$$(u-v)|q'(u)| \leqslant MB\left(\frac{u+v}{u-v}\right)^c (u-v)^c < 2MB(u-v)^c,$$

or

$$|q'(u)| < 2MB(u-v)^{c-1}.$$

Inserting this latter estimate into (4.3) yields

$$\begin{aligned} |q(u)| &\leqslant 2M^2Bvu^c \int_u^\infty t^{-c-1}t^{c-1}\, dt = \left(\frac{2M^2v}{u}\right) Bu^c \\ &< \tfrac{1}{2}Bu^c \quad \text{if} \quad u > 4M^2 v. \end{aligned} \tag{4.5}$$

Thus by (4.4) and (4.5), if in (4.2) we take $u_0 > v \max\left(1 + 2c/\log 2, 4M^2\right)$, then $q(u) = 0$ for $u > u_0$. Now rewrite (0.3) as

$$\left(u^{-\alpha} q(u)\right)' = u^{-\alpha-1} \sum_{j=1}^m \alpha_j q(u+v_j). \tag{4.6}$$

Since $v_j > 0$ for $j \geqslant 1$, it follows that $q(u) = 0$ for $u > 0$.

Case $0 \leqslant c < 1$**.** As in the previous case, from (0.3), we get

$$(u-v)|q'(u)| \leqslant u|q'(u)| \leqslant \sum_{j=0}^m |\alpha_j||q(u+v_j)| \leqslant MB(u+v)^c,$$

which implies that

$$|q'(u)| \leqslant MB\left(\frac{u+v}{u-v}\right)(u+v)^{c-1} \leqslant 2MB(u+v)^{c-1}, \qquad u > 3v. \tag{4.7}$$

Now use this latter estimate in the integral (4.3). We obtain

$$\begin{aligned} |q(u)| &\leqslant (2M^2v)Bu^c \int_u^\infty t^{-c-1}t^{c-1}\,dt = \left(\frac{2M^2v}{u}\right)Bu^c \\ &< \tfrac{1}{2}Bu^c, \quad \text{if} \quad u > 4M^2v. \end{aligned} \tag{4.8}$$

Thus by (4.7) and (4.8), if in (4.2) we take $u_0 > v\max(3, 4M^2)$, then $q(u) = 0$ for $u > u_0$. Then (4.6) implies $q(u) = 0$ for $u > 0$.

Case $c < 0$. From (0.3), we get

$$u|q'(u)| \leqslant \sum_{j=0}^{m} |\alpha_j||q(u+v_j)| \leqslant MBu^c.$$

Using this latter estimate in the integral (4.3), we get

$$\begin{aligned} |q(u)| &\leqslant (M^2v)Bu^c \int_u^\infty t^{-c-1}t^{c-1}\,dt = \left(\frac{M^2v}{u}\right)Bu^c \\ &< \tfrac{1}{2}Bu^c \quad \text{if} \quad u > 2M^2v. \end{aligned}$$

Therefore, if in (4.2) we take $u_0 > 2M^2v$, then $q(u) = 0$ for $u > u_0$. Then (4.6) implies $q(u) = 0$ for $u > 0$.

5. Another Uniqueness Result

In this section, our aim is to replace the normalizing condition (2.0) in Theorem 3 with the weaker condition that the solution to (0.3) be polynomially bounded.

Theorem 4. *Suppose that a function q satisfies the difference differential equation*

$$sq'(s) = \sum_{j=0}^{m} \alpha_j q(s+v_j), \qquad \Re(s) > 0 \tag{5.1}$$

and the additional requirement that for some fixed real number r,

$$q(s) = O(|s|^r), \qquad |s| \to \infty, \qquad \Re(s) > 0. \tag{5.2}$$

Let

$$q^*(s) := \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftarrow\!\!\!\circlearrowleft} z^{-\beta-1} \exp\left\{ sz - \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \right\} dz, \qquad \Re(s) > 0. \tag{5.3}$$

Then q is a constant multiple of q^*.

Proof. Suppose that q satisfies (5.1) and (5.2). Let $c := \Re(\beta)$. Without loss of generality, we may assume that $r > c$. From (5.1),

$$q^{(n)}(s) = O(|s|^{r-n}) \tag{5.4}$$

for all non-negative integers n. Fix $n > r+2$, $a > 0$ and define

$$g(x) := \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xz} q^{(n)}(z)\, dz, \qquad x > 0. \tag{5.5}$$

Note that the definition (5.5) is independent of $a > 0$. By absolute convergence of the differentiated integral,

$$\begin{aligned}
g'(x) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xz} z q^{(n)}(z)\, dz \\
&= \frac{e^{xz} z q^{(n)}(z)}{2\pi i x}\bigg|_{a-i\infty}^{a+i\infty} - \frac{1}{2\pi i x} \int_{a-i\infty}^{a+i\infty} e^{xz} \left(z q^{(n)}(z) \right)' dz \\
&= -\frac{1}{2\pi i x} \int_{a-i\infty}^{a+i\infty} e^{xz} q^{(n)}(z)\, dz - \frac{1}{2\pi i x} \int_{a-i\infty}^{a+i\infty} e^{xz} z q^{(n+1)}(z)\, dz \\
&= -\frac{g(x)}{x} - \frac{1}{2\pi i x} \int_{a-i\infty}^{a+i\infty} e^{xz} \left\{ \sum_{j=0}^{m} \alpha_j q^{(n)}(z+v_j) - n q^{(n)}(z) \right\} dz \\
&= \frac{n-1}{x} g(x) - \frac{1}{x} \sum_{j=0}^{m} \alpha_j \frac{1}{2\pi i} \int_{a+v_j-i\infty}^{a+v_j+i\infty} e^{x(z-v_j)} q^{(n)}(z)\, dz \\
&= \frac{n-1}{x} g(x) - \frac{1}{x} g(x) \sum_{j=0}^{m} \alpha_j e^{-v_j x}
\end{aligned}$$

Solving the separable first order ordinary differential equation for g, it follows that

$$g(x) = C \frac{(-1)^n}{\Gamma(-\beta)} x^{n-\beta-1} \exp\left\{ \sum_{j=0}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \right\},$$

where C is an arbitrary constant.

Claim.

$$q^{(n)}(s) = \int_0^\infty e^{-sx} g(x)\,dx, \qquad \Re(s) > 0. \tag{5.6}$$

Proof of Claim. Let $\Re(s) > a$. We have

$$\begin{aligned}\int_0^\infty e^{-sx} g(x)\,dx &= \int_0^\infty e^{-sx} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xz} q^{(n)}(z)\,dz\,dx \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} q^{(n)}(z) \int_0^\infty e^{-x(s-z)}\,dx\,dz \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{q^{(n)}(z)}{s-z}\,dz, \end{aligned} \tag{5.7}$$

where the interchange of integration order is justified by absolute convergence. Now let $b > \Re(s)$, $T > 0$ and integrate clockwise around the rectangle with corners $a \pm iT$, $b \pm iT$. By the residue theorem, it suffices to show that

$$\lim_{\substack{T\to\infty \\ b\to\infty}} I = \lim_{\substack{T\to\infty \\ b\to\infty}} \bar{I} = \lim_{\substack{T\to\infty \\ b\to\infty}} J = 0$$

where

$$I := \int_{a+iT}^{b+iT} \frac{q^{(n)}(z)}{s-z}\,dz \quad \text{and} \quad J := \int_{b-iT}^{b+iT} \frac{q^{(n)}(z)}{s-z}\,dz$$

and $\bar{I}$ denotes the complex conjugate of I. Now from (5.5) and the choice $n > r+2$,

$$|I| \leqslant \int_a^b \frac{|q^{(n)}(x+iT)|}{|s-x-iT|}\,dx \ll \frac{1}{T} \int_a^b \frac{dx}{x^2+T^2} \leqslant \frac{1}{T} \int_0^\infty \frac{dx}{x^2+T^2}.$$

Thus, $|I| \to 0$ as $T, b \to \infty$. Similarly, $|\bar{I}| \to 0$ as $T, b \to \infty$. Next,

$$|J| \leqslant \int_{-T}^{T} \frac{|q^{(n)}(b+it)|}{|s-b-it|}\,dt \ll \frac{1}{|s-b|} \int_{-T}^{T} \frac{dt}{b^2+t^2} \leqslant \frac{1}{|s-b|} \int_{-\infty}^{\infty} \frac{dt}{b^2+t^2}.$$

Thus, $|J| \to 0$ as $T, b \to \infty$. It follows from Cauchy's Theorem that

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{q^{(n)}(z)}{s-z}\,dz = q^{(n)}(s). \tag{5.8}$$

Since $a > 0$ was arbitrary, comparing (5.8) with (5.7) shows that the claim is now proved.

Continuing with the proof of Theorem 4, we rewrite (5.6) in terms of a loop integral of the form (2.1). Thus (using the Hankel form of the Laplace transform as in (1.1)) we get

$$q^{(n)}(s) = C\frac{\Gamma(\beta+1)}{2\pi i}\int z^{n-\beta-1}\exp\left\{sz - \sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}dz,$$

which implies that

$$\begin{aligned} q(s) &= C\frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}\exp\left\{sz - \sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}dz + p_n(s) \\ &= Cq^*(s) + p_n(s) \end{aligned}$$

where $p_n(s)$ is a polynomial in s of degree less than n. If $p_n = 0$, we're done. Otherwise, we argue as follows. Differentiating under the integral sign shows that $q^*(s)$ satisfies (5.1). It follows that $p_n(s) = q(s) - Cq^*(s)$ must also satisfy (5.1). For $u > 0$, it is clear that $q^*(u)$ and $p_n(u)$ both satisfy (2.0). By Theorem 3, p_n is a multiple of q^* and the proof of Theorem 4 is complete.

6. Behaviour at Zero and Infinity

Recall from (3.1) that if β is a non-negative integer, then the solution (2.1) reduces to the polynomial

$$Q_\beta(u) = \left(\frac{\partial}{\partial z}\right)^\beta\bigg|_{z=0}\exp\left\{uz - \sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}$$

of §3. This suggests that for general values of β, it may be profitable to consider $q(u)$ given by (2.1) as a fractional derivative. Thus, in a sense soon to be made precise,

$$q(u) \sim \left(\frac{\partial}{\partial z}\right)^\beta\bigg|_{z=0}\exp\left\{uz - \sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}.$$

For example, applying Leibniz's rule to the above yields the formal expansion

$$\begin{aligned} q(u) &\sim \sum_{n=0}^{\infty}\binom{\beta}{n}\left(\frac{\partial}{\partial z}\right)^n\bigg|_{z=0}\exp\left\{-\sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}\left(\frac{\partial}{\partial z}\right)^{\beta-n}\bigg|_{z=0}e^{uz} \\ &= \sum_{n=0}^{\infty}\binom{\beta}{n}Q_n(0)u^{\beta-n}, \qquad (6.1) \end{aligned}$$

which, in view of our previous remarks, gives a true equality when β is a non-negative integer. Here, $Q_n(0)$ is given by (3.2) and the fractional derivative of e^{uz} is given, as usual, by

$$\left(\frac{\partial}{\partial z}\right)^{\beta-n}\Big|_{z=0} e^{uz} = \frac{\Gamma(\beta+1-n)}{2\pi i}\int z^{n-\beta-1}e^{uz}\,dz = u^{\beta-n},$$

(cf. remarks following (1.1) for the case $\beta - n$ is a negative integer). Observe that the $n = 0$ term in (6.1) gives the known asymptotic formula $q(u) \sim u^{\beta}$ as $u \to \infty$. It turns out that (6.1) is a valid asymptotic expansion of q for u-values tending to positive infinity (cf. Iwaniec [12] and Wheeler [16, 17]).

Theorem 5. *Let q be given by (2.1) and let $Q_n(0)$ be given by (3.2). Then the asymptotic expansion*

$$q(u) \sim \sum_{n=0}^{\infty}\binom{\beta}{n}Q_n(0)u^{\beta-n}, \qquad u \to \infty,$$

is valid.

Proof. We have

$$q(u) = \frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}\exp\left\{uz - \sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}dz.$$

Let $n > \Re(\beta)$ and define R_n by

$$\exp\left\{-\sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} = \sum_{k=0}^{n-1}\frac{z^k}{k!}Q_k(0) + R_n(-z).$$

Then by Hankel's formula (1.1),

$$\begin{aligned} q(u) &= \sum_{k=0}^{n-1}\frac{Q_k(0)}{k!}\cdot\frac{\Gamma(\beta+1)}{2\pi i}\int z^{k-\beta-1}e^{uz}\,dz + \frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}e^{uz}R_n(-z)\,dz \\ &= \sum_{k=0}^{n-1}\frac{Q_k(0)}{k!}\cdot\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-k)}u^{\beta-k} + \frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}e^{uz}R_n(-z)\,dz \\ &= \sum_{k=0}^{n-1}\binom{\beta}{k}Q_k(0)u^{\beta-k} + \frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}e^{uz}R_n(-z)\,dz. \end{aligned}$$

Let $c := \Re(\beta)$. It remains to show that

$$\frac{\Gamma(\beta+1)}{2\pi i}\int_{\leftrightarrows\circlearrowleft} z^{-\beta-1}e^{uz}R_n(-z)\,dz = O(u^{c-n}), \qquad u\to\infty.$$

We have $|R_n(-z)| \ll |z|^n$ as $|z|\to 0+$ and $|R_n(-z)| \ll \exp(o(|z|))$ as $z\to-\infty$. Since $n > \Re(\beta)$, we may collapse the contour onto the negative real axis. Familiar calculations produce

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2\pi i}\int_{\leftrightarrows\circlearrowleft} z^{-\beta-1}e^{uz}R_n(-z)\,dz\\
&\ll \int_0^\infty x^{-c-1}e^{-ux}|R_n(x)|\,dx\\
&\ll \int_0^1 x^{-c-1}e^{-ux}|R_n(x)|\,dx + \int_1^\infty x^{-\beta-1}e^{-ux}\left|\exp\left\{\sum_{j=0}^m \alpha_j\int_0^x \frac{1-e^{-v_jt}}{t}\,dt\right\}\right|dx\\
&\qquad + \int_1^\infty x^{-c-1}e^{-ux}\left|\sum_{k=0}^{n-1}\frac{(-x)^k}{k!}Q_k(0)\right|dx\\
&\ll \int_0^1 x^{n-c-1}e^{-ux}\,dx + \int_1^\infty e^{-ux}\,dx + \int_1^\infty x^{n-c-1}e^{-ux}\,dx\\
&= \frac{\Gamma(n-c)}{u^{n-c}} + O\left(e^{-u}\right)\\
&\ll u^{c-n}, \qquad u\to\infty.
\end{aligned}$$

The implied constant may depend on all parameters except u.

It is interesting to deduce some properties of the coefficients $Q_n(0)$ of the asymptotic expansion provided by Theorem 5. Rewriting (3.2) in the form

$$\exp\left\{-\sum_{j=0}^m \alpha_j\int_0^z \frac{e^{v_jt}-1}{t}\,dt\right\} = \sum_{n=0}^\infty \frac{z^n}{n!}Q_n(0), \tag{6.2}$$

it is immediate that $Q_0(0)=1$. For positive integers n, we have the following recurrence formula which gives $Q_n(0)$ in terms of $Q_0(0)$, $Q_1(0)$, $\ldots$, $Q_{n-1}(0)$.

Proposition 6.1. *If n is a positive integer and $Q_n(0)$ is given by (6.2), then*

$$nQ_n(0) = -\sum_{k=0}^{n-1}\binom{n}{k}Q_k(0)\sum_{j=0}^m \alpha_j v_j^{n-k}.$$

Proof. Applying the operator $z \cdot d/dz$ to (6.2), one obtains

$$\sum_{n=0}^{\infty} nQ_n(0)\frac{z^n}{n!} = \exp\left\{-\sum_{j=0}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\}\sum_{j=0}^{m}\alpha_j\left(1-e^{v_j z}\right).$$

The result now follows on comparing coefficients of $z^n/n!$. For concreteness, the first few values are listed below.

$$Q_0(0) = 1, \quad Q_1(0) = -\sum_{j=0}^{m}\alpha_j v_j, \quad Q_2(0) = -\frac{1}{2}\sum_{j=0}^{m}\alpha_j v_j^2 + \left(\sum_{j=0}^{m}\alpha_j v_j\right)^2.$$

Next, we analyze the behaviour of $Q_n(0)$ for large n. Since the generating function (6.2) is entire, it follows by Hadamard's root test that $\limsup_{n\to\infty}\sqrt[n]{|Q_n(0)/n!|} = 0$. However, since this information does not reveal how quickly or how slowly $|Q_n(0)/n!|$ tends to zero, we seek a concrete upper bound. (cf. Bradley [3] in which the special case corresponding to equation (0.1) is addressed.)

Theorem 6. *Let $M := m \times \max_{1\leqslant j\leqslant m}|\alpha_j|$, and $v := v_m = \max_j v_j$. Then, for all positive integers n, we have*

$$\left|\frac{Q_n(0)}{n!}\right| \leqslant \left(\frac{ev}{\log(1+n/M)}\right)^n,$$

where again, $Q_n(0)$ is given by (6.2).

Proof. We begin by observing that since $v_0 = 0$, the integral in (6.2) with $j = 0$ vanishes, and we can start the sum with $j = 1$. For any $r > 0$, Cauchy's inequality applied to (6.2) then gives

$$\left|\frac{Q_n(0)}{n!}\right| \leqslant r^{-n}\max_{|z|=r}\left|\exp\left\{-\sum_{j=1}^{m}\alpha_j\int_0^z\frac{e^{v_j t}-1}{t}\,dt\right\}\right|.$$

Now on the circle $\{z \in \mathbf{C} : |z| = r\}$,

$$\left|\exp\left\{-\sum_{j=1}^{m}\alpha_j\int_0^z\frac{e^{v_j t}-1}{t}\,dt\right\}\right| \leqslant \exp\left\{\sum_{j=1}^{m}|\alpha_j|\int_0^r\frac{e^{v_j t}-1}{t}\,dt\right\}.$$

Thus

$$\left|\frac{Q_n(0)}{n!}\right| \leqslant \inf_{r>0} r^{-n}\exp\left\{M\int_0^r\frac{e^{vt}-1}{t}\,dt\right\}. \tag{6.3}$$

Minimizing with respect to $r > 0$, we find on taking the logarithmic derivative that $Mr^{-1}(e^{rv}-1) - r^{-1}n = 0$ so that for $n \geqslant 1$, $r = v^{-1}\log(1+n/M)$. If we use the fact that

$$\int_0^r \frac{e^{vt}-1}{t}\,dt = \sum_{n=1}^{\infty} \frac{(rv)^n}{n!\,n} \leqslant e^{rv}-1,$$

then (6.3) with optimal r yields

$$\left|\frac{Q_n(0)}{n!}\right| \leqslant \left(\frac{ev}{\log(1+n/M)}\right)^n,$$

as required.

The asymptotic expansion of Theorem 5 describes the behaviour of $q(u)$ as $u \to \infty$. The next task is to elucidate the behaviour of $q(u)$ as $u \to 0+$. Our result extends the treatment of the $m = 1$ case in Wheeler [16].

Theorem 7. *Suppose that q as given by (2.1) satifies the difference differential equation (0.3). For convenience, put $\alpha := \alpha_0$ and $\delta := \alpha_1 + \alpha_2 + \cdots + \alpha_m = \beta - \alpha$.*
i) If $\alpha = 0$ and $\beta \neq 0, 1, 2, \ldots$, then

$$q(u) \sim -\left(\frac{e^{\beta\gamma}}{\Gamma(-\beta)} \prod_{j=1}^{m} v_j^{\alpha_j}\right) \log u \qquad u \to 0+.$$

ii) If $\Re(\alpha) < 0$ and $\beta \neq 0, 1, 2, \ldots$, then

$$q(u) \sim u^{\alpha} e^{\delta\gamma} \frac{\Gamma(-\alpha)}{\Gamma(-\beta)} \prod_{j=1}^{m} v_j^{\alpha_j}, \qquad u \to 0+.$$

iii) If $\Re(\alpha) > 0$ or β is a non-negative integer, then $\lim_{u\to 0+} q(u)$ exists and is given by

$$\alpha q(0) = -\sum_{j=1}^{m} \alpha_j q(v_j).$$

Remark. If β is a non-negative integer then rearranging iii) gives another proof of (3.6).

Proof i). Since $\alpha_0 = \alpha = v_0 = 0$ and $v_j > 0$ for $j \geqslant 1$, we may integrate (0.3) as follows:

$$\begin{aligned} q(u) &= q(1) + \sum_{j=1}^{m} \alpha_j \int_1^u q(t+v_j)\,\frac{dt}{t} \\ &= q(1) + \left(\sum_{j=1}^{m} \alpha_j q(v_j)\right) \log u + \sum_{j=1}^{m} \alpha_j \int_1^u \frac{q(t+v_j) - q(v_j)}{t}\,dt \\ &= \left(\sum_{j=1}^{m} \alpha_j q(v_j)\right) \log u + O(1), \qquad u \to 0+. \end{aligned}$$

If we now use (2.1) and employ the reflection formula for the gamma function and the relationship (1.2) between exponential integrals, then we obtain

$$\begin{aligned}
\sum_{j=1}^{m}\alpha_j q(v_j) &= \frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}\Big(\sum_{j=1}^{m}\alpha_j e^{v_j z}\Big)\exp\Big\{-\sum_{j=1}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\Big\}\,dz \\
&= -\frac{\Gamma(\beta+1)}{2\pi i}\int \Big(z^{-\beta}\exp\Big\{-\sum_{j=1}^{m}\alpha_j\int_0^z \frac{e^{v_j t}-1}{t}\,dt\Big\}\Big)'\,dz \\
&= -\frac{\Gamma(\beta+1)}{2\pi i}\lim_{x\to\infty}\big(e^{-\pi i\beta}-e^{\pi i\beta}\big)\,x^{-\beta}\exp\Big\{\sum_{j=1}^{m}\alpha_j\int_0^x \frac{1-e^{-v_j t}}{t}\,dt\Big\} \\
&= \frac{\Gamma(\beta+1)\sin(\pi\beta)}{\pi}\lim_{x\to\infty}x^{-\beta}\exp\Big\{\sum_{j=1}^{m}\alpha_j\big(\log(v_j x)+\gamma+\mathrm{E}_1(v_j x)\big)\Big\} \\
&= -\frac{e^{\beta\gamma}}{\Gamma(-\beta)}\prod_{j=1}^{m}v_j^{\alpha_j}. \qquad (6.4)
\end{aligned}$$

Proof ii). First, suppose that $\Re(\beta)<0$. Then the Laplace transform representation (2.6) of Corollary 2.4 is valid, and as $u\to 0+$ we have

$$\begin{aligned}
q(u) &= \frac{1}{\Gamma(-\beta)}\int_0^\infty \Big(\frac{y}{u}\Big)^{-\beta-1}\exp\Big\{-y+\sum_{j=1}^{m}\alpha_j\int_0^{v_j y/u}\frac{1-e^{-t}}{t}\,dt\Big\}\frac{dy}{u} \\
&\sim \frac{1}{\Gamma(-\beta)}\int_0^\infty \Big(\frac{y}{u}\Big)^{-\beta-1}\exp\Big\{-y+\sum_{j=1}^{m}\alpha_j\big(\log(v_j y/u)+\gamma+\mathrm{E}_1(v_j y/u)\big)\Big\}\frac{dy}{u} \\
&= \frac{e^{\delta\gamma}}{\Gamma(-\beta)}\int_0^\infty \Big(\frac{y}{u}\Big)^{-\alpha-1}\exp\Big\{-y+\sum_{j=1}^{m}\alpha_j\big(\log(v_j)+\mathrm{E}_1(v_j y/u)\big)\Big\}\frac{dy}{u} \\
&\sim \frac{u^\alpha e^{\delta\gamma}}{\Gamma(-\beta)}\prod_{j=1}^{m}v_j^{\alpha_j}\int_0^\infty y^{-\alpha-1}e^{-y}\,dy \\
&= u^\alpha e^{\delta\gamma}\frac{\Gamma(-\alpha)}{\Gamma(-\beta)}\prod_{j=1}^{m}v_j^{\alpha_j}. \qquad (6.5)
\end{aligned}$$

Now let n be a non-negative integer such that $0\leqslant\Re(\beta)<n$. Differentiating the loop integral representation (2.1) of Proposition 2.2 n times with respect to u shows that $q^{(n)}(u)$ is equal to $\beta(\beta-1)\cdots(\beta-n+1)$ times $q(u,\alpha-n,\beta-n)$, the q function which has been altered by replacing α with $\alpha-n$ and β with $\beta-n$. By (6.5),

$$q^{(n)}(u)\sim\beta(\beta-1)\cdots(\beta-n+1)u^{\alpha-n}e^{\delta\gamma}\frac{\Gamma(n-\alpha)}{\Gamma(n-\beta)}\prod_{j=1}^{m}v_j^{\alpha_j}. \qquad (6.6)$$

Now integrate (6.6) n times with respect to u and apply the functional equation for the gamma function. There is a polynomial p_n of degree less than n for which, as $u \to 0+$,

$$\begin{aligned} q(u) &\sim \frac{\beta(\beta-1)\cdots(\beta-n+1)}{\alpha(\alpha-1)\cdots(\alpha-n+1)} u^\alpha e^{\delta\gamma} \frac{\Gamma(n-\alpha)}{\Gamma(n-\beta)} \prod_{j=1}^m v_j^{\alpha_j} + p_n(u) \\ &= u^\alpha e^{\delta\gamma} \frac{\Gamma(-\alpha)}{\Gamma(-\beta)} \prod_{j=1}^m v_j^{\alpha_j} + p_n(u). \end{aligned}$$

The conclusion of ii) now follows.

Proof iii). Recall that in the proof of ii), we used the representation

$$q'(u) = \beta q(u, \alpha-1, \beta-1),$$

where $q(u, \alpha-1, \beta-1)$ is $q(u)$ with α replaced by $\alpha-1$ and β replaced by $\beta-1$. The difference differential equation (0.3) gives

$$\alpha q(u,\alpha,\beta) = u\beta q(u,\alpha-1,\beta-1) - \sum_{j=1}^m \alpha_j q(u+v_j,\alpha,\beta). \tag{6.7}$$

Now if β is a non-negative integer, then q is the polynomial of §3 given by (3.1). In particular, if $\beta = 0$, then $q(u) = 1$ for all u and iii) is trivially satisfied. If β is a positive integer, then every term in (6.7) is a polynomial, and iii) follows after letting $u = 0$.

Next, suppose that $\Re(\alpha) > 0$ and $\beta \in \mathbf{C}$. From the difference differential equation (0.3) and the contour integral representation (2.1), for $u > 0$ we have

$$uq'(u) = \frac{\Gamma(\beta+1)}{2\pi i} \int z^{-\beta-1} \sum_{r=0}^m \alpha_r \exp\left\{(u+v_r)z - \sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} dz. \tag{6.8}$$

However, when $\Re(\alpha) > 0$, the representation (6.8) is also valid at $u = 0$. This is due to the fact that for z on the contour and $|z|$ large, we have, in view of (1.2),

$$\begin{aligned} f(z) &:= \exp\left\{-\sum_{j=0}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\ &= \exp\left\{-\sum_{j=1}^m \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\,dt\right\} \\ &= \exp\left\{\sum_{j=1}^m \alpha_j \big(\log(-v_j z) + \gamma + \mathrm{E}_1(-v_j z)\big)\right\} \\ &= O\big(|z|^{\beta-\alpha}\big). \end{aligned} \tag{6.9}$$

It follows that

$$
\begin{aligned}
\lim_{u\to 0} uq'(u) &= \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta-1} f(z) \sum_{j=0}^{m} \alpha_j e^{v_j z}\, dz \\
&= \beta \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta-1} f(z)\, dz - \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta-1} f(z) \sum_{j=0}^{m} \alpha_j \left(1 - e^{v_j z}\right) dz \\
&= \beta \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta-1} f(z)\, dz - \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta} f'(z)\, dz. \qquad (6.10)
\end{aligned}
$$

Take the second integral in (6.10) and integrate by parts. The integrated term will vanish due to (6.9) and our assumption that $\Re(\alpha) > 0$. Then (6.10) becomes

$$
\lim_{u\to 0} uq'(u) = \beta \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta-1} f(z)\, dz - \beta \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\circlearrowleft} z^{-\beta-1} f(z)\, dz = 0.
$$

By the difference differential equation (0.3), it follows that

$$
0 = \sum_{j=0}^{m} \alpha_j q(v_j),
$$

or, since $\alpha := \alpha_0$ and $v_0 = 0$,

$$
\alpha q(0) = -\sum_{j=1}^{m} \alpha_j q(v_j),
$$

as required.

7. An Operator Representation

We begin this section with an informal argument which should help motivate what follows. Let $f(z)$ be a formal power series in z and let $D = d/du$. Since $D^n e^{uz} = z^n e^{uz}$ for all non-negative integers n, it follows by linearity that the equation $f(D)e^{uz} = f(z)e^{uz}$ holds, at least in the formal sense. If we now apply this observation to the contour integral representation (2.1) with

$$
f(z) := \exp\left\{ -\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t} - 1}{t}\, dt \right\},
$$

we obtain

$$\begin{aligned} q(u) &= \frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}\exp\Big\{-\sum_{j=0}^{m}\alpha_j\int_0^D \frac{e^{v_j t}-1}{t}\,dt\Big\}e^{uz}\,dz \\ &= \exp\Big\{-\sum_{j=0}^{m}\alpha_j\int_0^D \frac{e^{v_j t}-1}{t}\,dt\Big\}\frac{\Gamma(\beta+1)}{2\pi i}\int z^{-\beta-1}e^{uz}\,dz \\ &= e^{\mathscr{L}}u^{\beta}, \end{aligned} \tag{7.1}$$

where the operator $\mathscr{L}$ is given by

$$\mathscr{L} := -\sum_{j=0}^{m}\alpha_j L(v_j), \tag{7.2}$$

and the operators $L(\lambda)$ are given by

$$L(\lambda) := \int_0^D \frac{e^{\lambda t}-1}{t}\,dt = \sum_{n=1}^{\infty}\frac{(\lambda D)^n}{n!\,n}. \tag{7.3}$$

Pulling the differential operator outside the integral requires justification, and we shall do this shortly. But for the moment, a few remarks about (7.1) are in order. Expanding $e^{\mathscr{L}}$ in powers of D, we formally obtain the expression

$$q(u) \sim \sum_{n=0}^{\infty} Q_n(0)\frac{D^n}{n!}u^{\beta} = \sum_{n=0}^{\infty}\binom{\beta}{n}Q_n(0)u^{\beta-n} \tag{7.4}$$

in agreement with Theorem 5. In particular, the $n = 0$ term gives the known asymptotic formula $q(u) \sim u^{\beta}$ as $u \to \infty$. Furthermore, when $\beta = n$ is a non-negative integer (7.4) terminates and gives the polynomial of (3.1).

Next, we point out that the differential operator $L(\lambda)$ can be recast in the form of an integral operator. We have

$$L(\lambda) = \int_0^1 \frac{e^{\lambda t D}-1}{t}\,dt.$$

Now if f is analytic in a disk centred at u with radius $r > 0$, and $|t\lambda| < r$, then Taylor's theorem gives

$$e^{\lambda t D}f(u) = \sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}D^n f(u) = \sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}f^{(n)}(u) = f(u+\lambda t).$$

Thus for those functions f which are analytic in a disk centred at u with radius $r > |\lambda|$,

$$L(\lambda)f(u) = \int_0^1 \frac{f(u+\lambda t)-f(u)}{t}\,dt. \tag{7.5}$$

Now the integral in (7.5) makes sense if f is integrable on $[u, u+\lambda]$ and for some $\varepsilon > 0$, we have $|f(u+\lambda t) - f(u)| \ll t^{\varepsilon}$ as $t \to 0+$. *For such f, we henceforth define $L(\lambda)f$ by (7.5).* The new operator $L(\lambda)$, a Hellinger integro-differential operator, applies to a larger class of functions than merely those functions which are analytic in a suitably large disk centred at u. Thus there is no need to view $L(\lambda)$ as a power series in D in order to determine $L(\lambda)f$. In the case of interest, $f(u) = u^{\beta}$ and (7.1) becomes

$$q(u) = e^{\mathscr{L}} u^{\beta} = \sum_{n=0}^{\infty} \frac{\mathscr{L}^n}{n!} u^{\beta} \tag{7.6}$$

and so it makes sense to study the iterated integral operator $\mathscr{L}^n$. We shall take this up in §8. The remainder of §7 is devoted to proving the operator series representation (7.6) rigorously, followed by an analysis of the error created by truncating the series to n terms.

Theorem 8. *Let $D := d/du$, let the operators $L(\lambda)$ be given by (7.5), and let $\mathscr{L}$ be given by (7.2). Then the solution (2.1) of the difference differential equation (0.3) has the operator representation*

$$q(u) = e^{\mathscr{L}} u^{\beta} = \sum_{n=0}^{\infty} \frac{\mathscr{L}^n}{n!} u^{\beta}.$$

Remark. In the boundary case $m = 0$, $\mathscr{L} = 0$ and we recover the solution $q(u) = u^{\beta}$.

Proof. Fix $u > 0$ and consider the function of the complex variable w defined by

$$h(w) := \frac{\Gamma(\beta+1)}{2\pi i} \int_{\Leftarrow\!\circlearrowleft} z^{-\beta-1} \exp\left\{ uz - w \sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t} - 1}{t}\, dt \right\} dz.$$

It's not hard to see that h is entire, and so

$$q(u) = h(1) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!}. \tag{7.7}$$

It remains to show that $h^{(n)}(0) = \mathscr{L}^n u^{\beta}$ for each non-negative integer n. When $n = 0$, we have

$$h(0) = \frac{\Gamma(\beta+1)}{2\pi i} \int_{\Leftarrow\!\circlearrowleft} z^{-\beta-1} e^{uz}\, dz = u^{\beta} = \mathscr{L}^0 u^{\beta}.$$

Suppose now that $h^{(k)}(0) = \mathscr{L}^k u^\beta$ for all $0 \leqslant k \leqslant n-1$. Then we have, by induction,

$$\begin{aligned}
h^{(n)}(0) &= \frac{\Gamma(\beta+1)}{2\pi i} \int_{\rightleftarrows\!\circlearrowleft} z^{-\beta-1} e^{uz} \Big\{ -\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}^n dz \\
&= \frac{\Gamma(\beta+1)}{2\pi i} \int_{\rightleftarrows\!\circlearrowleft} z^{-\beta-1} e^{uz} \Big\{ \sum_{j=0}^{m} \alpha_j \int_0^1 \frac{1-e^{v_j tz}}{t}\, dt \Big\} \\
&\qquad\qquad \times \Big\{ -\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}^{n-1} dz \\
&= \int_0^1 \frac{\Gamma(\beta+1)}{2\pi i} \int_{\rightleftarrows\!\circlearrowleft} z^{-\beta-1} e^{uz} \Big\{ -\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}^{n-1} \\
&\qquad\qquad \times \sum_{j=0}^{m} \alpha_j \left(1-e^{v_j zt}\right) dz\, \frac{dt}{t} \\
&= \int_0^1 \Big\{ \mathscr{L}^{n-1} u^\beta \sum_{j=0}^{m} \alpha_j - \mathscr{L}^{n-1} \sum_{j=0}^{m} \alpha_j \left(u+v_j t\right)^\beta \Big\} \frac{dt}{t} \\
&= \sum_{j=0}^{m} \alpha_j \int_0^1 \Big\{ \mathscr{L}^{n-1} u^\beta - \mathscr{L}^{n-1} \left(u+v_j t\right)^\beta \Big\} \frac{dt}{t} \\
&= \mathscr{L}^{n-1} \sum_{j=0}^{m} \alpha_j \int_0^1 \frac{u^\beta - \left(u+v_j t\right)^\beta}{t}\, dt \\
&= \mathscr{L}^{n-1} \Big\{ -\sum_{j=0}^{m} \alpha_j L(v_j) \Big\} u^\beta \\
&= \mathscr{L}^n u^\beta .
\end{aligned}$$

Having proved the representation (7.6), it is natural to ask how rapidly the series of $\mathscr{L}$-iterates converges to the function q. To this end, we prove the following

Theorem 9. *Let $c := \Re(\beta)$ and let n be a non-negative integer satisfying $n > c+1$. Define $M := m \times \max_{1\leqslant j\leqslant m} |\alpha_j|$, $v := v_m = \max_j v_j$, $d := \exp(\gamma + \mathrm{E}_1(v))$, and*

$$S_n := q(u) - \sum_{k=0}^{n-1} \frac{\mathscr{L}^k}{k!} u^\beta, \qquad u > 0.$$

Then as $n \to \infty$, we have

$$S_n \ll \frac{(Mv)^n}{n!} e^{Mv} + u^c \left(\frac{dMv}{u}\right) \left(\frac{u \log n}{n}\right)^{n(1-1/\log n)} .$$

The implied $\ll$ constant depends at most on β.

Proof. We have

$$\begin{aligned}
S_n &= \sum_{k=n}^{\infty} \frac{\mathscr{L}^k}{k!}\, u^\beta = \sum_{k=n}^{\infty} \frac{h^{(k)}(0)}{k!} \\
&= \sum_{k=n}^{\infty} \frac{1}{k!} \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\!\bigcirc} z^{-\beta-1} e^{uz} \Big\{ -\sum_{j=0}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}^k dz \\
&= \frac{\Gamma(\beta+1)}{2\pi i} \int_{\leftrightarrows\!\bigcirc} z^{-\beta-1} e^{uz} \sum_{k=n}^{\infty} \frac{1}{k!} \Big\{ -\sum_{j=1}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}^k dz.
\end{aligned}$$

We have omitted the $j = 0$ term from the inner sum because it vanishes (recall that $v_0 = 0$). Since $n > \Re(\beta)$ and

$$\sum_{k=n}^{\infty} \frac{1}{k!} \Big\{ -\sum_{j=1}^{m} \alpha_j \int_0^z \frac{e^{v_j t}-1}{t}\, dt \Big\}^k \ll |z|^n$$

as $z \to 0$, we may collapse the contour onto the real half-line, obtaining

$$S_n = \frac{1}{\Gamma(-\beta)} \int_0^\infty x^{-\beta-1} e^{-ux} \sum_{k=n}^{\infty} \frac{1}{k!} \Big\{ \sum_{j=1}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \Big\}^k dx.$$

Let

$$\begin{aligned}
I_n &:= \int_0^1 x^{-\beta-1} e^{-ux} \sum_{k=n}^{\infty} \frac{1}{k!} \Big\{ \sum_{j=1}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \Big\}^k dx, \\
J_n &:= \int_1^\infty x^{-\beta-1} e^{-ux} \sum_{k=n}^{\infty} \frac{1}{k!} \Big\{ \sum_{j=1}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \Big\}^k dx,
\end{aligned}$$

so that $S_n = (I_n + J_n)/\Gamma(-\beta)$. Next, for $x > 0$,

$$\Big| \sum_{j=1}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \Big| \leqslant M \int_0^{vx} \frac{1-e^{-t}}{t}\, dt \leqslant Mvx.$$

It follows that

$$\begin{aligned}
\Big| \sum_{k=n}^{\infty} \frac{1}{k!} \Big\{ \sum_{j=1}^{m} \alpha_j \int_0^x \frac{1-e^{-v_j t}}{t}\, dt \Big\}^k \Big| &\leqslant \sum_{k=n}^{\infty} \frac{(Mvx)^k}{k!} \\
&\leqslant \frac{(Mvx)^n}{n!} \Big(1 + \frac{Mvx}{n+1} + \frac{(Mvx)^2}{(n+1)(n+2)} + \cdots \Big) \\
&\leqslant \frac{(Mvx)^n}{n!} e^{Mvx}.
\end{aligned}$$

If we use the latter inequality in I_n and the assumption $n > c+1$, there results

$$(7.8) \qquad |I_n| \leqslant \frac{(Mv)^n}{n!} \int_0^1 x^{n-c-1} e^{x(Mv-u)}\,dx \leqslant \frac{(Mv)^n}{n!} e^{Mv} \int_0^1 x^{n-c-1}\,dx \leqslant \frac{(Mv)^n}{n!} e^{Mv},$$

from which it follows immediately that $|I_n| \to 0$ as $n \to \infty$.

We now turn to J_n. For $x \geqslant 1$ we may apply (1.2) in the form

$$\int_0^x \frac{1-e^{-v_j t}}{t}\,dt \leqslant \int_0^x \frac{1-e^{-vt}}{t}\,dt = \log(vx) + \gamma + \mathrm{E}_1(v) = \log(dvx).$$

If we use the latter inequality in J_n, there results

$$(7.9) \qquad |J_n| \leqslant \int_1^\infty x^{-c-1} e^{-ux} \sum_{k=n}^\infty \frac{(M\log(dvx))^k}{k!}\,dx.$$

Now let $r > M$ be a free parameter to be chosen later. We have

$$\begin{aligned} \sum_{k=n}^\infty \frac{(M\log(dvx))^k}{k!} &= \sum_{k=n}^\infty \left(\frac{M}{r}\right)^k \frac{(r\log(dvx))^k}{k!} \leqslant \left(\frac{M}{r}\right)^n \sum_{k=0}^\infty \frac{(r\log(dvx))^k}{k!} \\ &= \left(\frac{M}{r}\right)^n (dvx)^r. \end{aligned}$$

Substitute the latter inequality into (7.9). There results

$$(7.10) \qquad |J_n| \leqslant \left(\frac{M}{r}\right)^n (dv)^r \int_1^\infty x^{r-c-1} e^{-ux}\,dx \leqslant \left(\frac{M}{r}\right)^n (dv)^r \frac{\Gamma(r-c)}{u^{r-c}},$$

from which it follows immediately that $|J_n| \to 0$ as $n \to \infty$. But much more can be said if r is chosen optimally so as to minimize the upper bound (7.10). On taking the logarithmic derivative, the optimal r is seen to satisfy

$$(7.11) \qquad n = r\{\psi(r-c) + \log(dv/u)\},$$

provided $r > M$. Here, as customary, $\psi = \Gamma'/\Gamma$. Inverting (7.11) and using the fact that $\psi(r-c) \sim \log r$ as $r \to \infty$, we find that r is approximately $n/\log n$. With this choice, (7.10) yields (after applying Stirling's formula)

$$\begin{aligned} |J_n| &\leqslant u^c M^n \left(\frac{dv}{u}\right)^{n/\log n} \left(\frac{\log n}{n}\right)^n \Gamma\left(\frac{n}{\log n} - c\right) \\ (7.12) \qquad &\ll u^c \left(\frac{dMv}{u}\right)^n \left(\frac{u\log n}{n}\right)^{n(1-1/\log n)}. \end{aligned}$$

Recall that $S_n = (I_n + J_n)/\Gamma(-\beta)$. Combining (7.8) with (7.12) completes the proof.

8. The Operators $L^n(\lambda)$

We now return to study the integral operators $L(\lambda)$ and their iterates in more detail. Recall that for $f \in L^1[u, u+\lambda]$ satisfying a Lipschitz condition at u, we may define

$$L(\lambda)f(u) = \int_0^1 \frac{f(u+\lambda t) - f(u)}{t}\, dt, \tag{8.1}$$

which is equivalent to

$$L(\lambda) = \int_0^1 \frac{e^{\lambda t D} - 1}{t}\, dt = \sum_{n=1}^{\infty} \frac{(\lambda D)^n}{n!\, n}, \qquad D := d/du \tag{8.2}$$

if f is analytic in a disc centred at u with radius $r > |\lambda|$. We remark that the representation (8.1) has occurred in important contexts outside sieve theory. For example,

$$\begin{aligned} L(\lambda)\log u &= \int_0^1 \frac{\log(u+\lambda t) - \log u}{t}\, dt = \int_0^1 \log(1+\lambda t/u)\, \frac{dt}{t} \\ &= \int_0^{-\lambda/u} \log(1-w)\, \frac{dw}{w} \\ &= -\operatorname{Li}_2\left(-\lambda/u\right), \end{aligned}$$

where

$$\operatorname{Li}_2(z) := -\int_0^z \log(1-w)\, \frac{dw}{w} = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1,$$

is the ubiquitous dilogarithm. We give some additional represention for $L(\lambda)$ and its iterates.

Proposition 8.1. *Fix $\lambda \in \mathbf{R}$ and let n be a positive integer. Suppose that $f \in C^{n+1}[u, u+\lambda]$. Then*

$$\begin{aligned} L^n(\lambda)f(u) &= \lambda^n \int_{[0,1]^n} f^{(n)}(u + \lambda t_1 + \lambda t_2 + \cdots + \lambda t_n) \prod_{k=1}^{n} \log(1/t_k)\, dt_k \\ &= \int_{[0,\lambda]^n} f^{(n)}(u + t_1 + t_2 + \cdots + t_n) \prod_{k=1}^{n} \log(\lambda/t_k)\, dt_k. \end{aligned}$$

Proof. The conditions on f permit us to integrate (8.1) by parts. Thus,

$$\begin{aligned} L(\lambda)f(u) &= \left\{f(u+\lambda t) - f(u)\right\}\log t\Big|_0^1 - \lambda\int_0^1 f'(u+\lambda t)\log t\, dt \\ &= \lambda\int_0^1 f'(u+\lambda t)\log(1/t)\, dt. \end{aligned}$$

The integrated term vanishes because

$$\lim_{t\to 0+} \frac{f(u+\lambda t)-f(u)}{t} \cdot t\log t = \lambda f'(u) \lim_{t\to 0+} t\log t = 0.$$

The general case is handled inductively.

Proposition 8.2. *Fix* $\lambda \in \mathbf{R}$ *and let* n *be a positive integer. Suppose that* $f \in C^{n+1}[u, u+\lambda]$. *Then*

$$L^n(\lambda)f(u) = \lambda^n \int_{[0,1]^n} \int_{[0,1]^n} f^{(n)}(u+\lambda s_1 t_1 + \lambda s_2 t_2 + \cdots + \lambda s_n t_n) \prod_{k=1}^{n} ds_k\, dt_k.$$

Proof. By definition,

$$L(\lambda)f(u) = \int_0^1 \frac{f(u+\lambda t)-f(u)}{t}\, dt = \int_0^1 \int_u^{u+\lambda t} f'(x)\, dx\, \frac{dt}{t} = \lambda \int_0^1 \int_0^t f'(u+\lambda r)\, dr\, \frac{dt}{t}$$
$$= \lambda \int_0^1 \int_0^1 f'(u+\lambda st)\, ds\, dt.$$

The general case is handled inductively.

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David M. Bradley,
Centre for Experimental and
Constructive Mathematics,
Simon Fraser University,
Burnaby B.C. V5A 1S6 Canada
dbradley@cecm.sfu.ca

Harold G. Diamond,
Department of Mathematics,
University of Illinois at
Urbana-Champaign,
1409 W. Green St.,
Urbana IL 61801 U.S.A.
diamond@math.uiuc.edu