On a Class Number Formula for Real Quadratic Number Fields

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Abstract

For an even Dirichlet character ψ, we obtain a formula for $L(1, \psi)$ in terms of a sum of Dirichlet L-series evaluated at $s = 2$ and $s = 3$ and a rapidly convergent numerical series involving the central binomial coefficients. We then derive a class number formula for real quadratic number fields by taking $L(s, \psi)$ to be the quadratic L-series associated with these fields.

1 Introduction

In [1], acceleration formulæ are derived for Catalan's constant $L(2, \chi_4)$; (here χ_4 is the non-principal Dirichlet character of modulus 4). In some of these formulæ $L(2, \chi_4)$ is given as the sum of two terms: one involving a rapidly convergent series and the other involving the natural logarithm of a unit in the ring of integers of a finite abelian field extension of the rational number field $\mathbb{Q}$. The existence of the logarithmic terms suggested to the authors that these terms should somehow be related to the values of Dirichlet L-series at the argument $s = 1$. This leads to the general question of whether or not there exist relations between the value of L-series at $s = 1$ and values of L-series at integer arguments larger than 1.

The purpose of this note is to exhibit such a relation between values of L-series. For an even Dirichlet character ψ, we obtain a formula for $L(1, \psi)$ in terms of a sum of Dirichlet series evaluated at $s = 2$ and $s = 3$ and a convergent numerical series involving powers of twice special values of the sine function divided by $\binom{2n}{n} n^3$. See Theorem 1 below for a precise statement. (It is perhaps interesting to notice that not much is known about number theoretic properties of the values of the L-series on the right-hand side of the formula given in this theorem.) We then deduce a class number formula for real quadratic number fields by letting ψ be the quadratic character associated with a real quadratic number field; see Corollary 1. This class number formula seems new to us and is perhaps an interesting curiosity.

To derive our results, we employ a formula of Zucker [5] that expresses

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{2n}{n} n^3}, \qquad |x| \le 2, \tag{1}$$

in terms of periodic zeta functions. Proposition 1 below shows how periodic zeta functions may be expressed in terms of Dirichlet L-series. Thus, we can rewrite (1) in terms of L-series values, thereby obtaining our result.

2 Preliminaries

Let m be a positive integer. We denote the group of Dirichlet characters of modulus m by $\hat{U}_m$. The Dirichlet L-series associated with $\chi \in \hat{U}_m$ is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Similarly, for real β we define the *periodic zeta function* (a special case of the Lerch transcendent) by

$$\Phi(s,\beta) = \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Let $\zeta_m = e^{2\pi i/m}$. Throughout, the sum over a complete set of residues modulo m is denoted by $\sum_{a \bmod m}$ and the sum over the positive integer divisors of m is denoted by $\sum_{d|m}$. Thus, Ramanujan's sum is

$$c_m(k) = \sum_{\substack{\nu \bmod m \\ (\nu,m)=1}} \zeta_m^{\nu k},$$

and likewise the Gaussian sum attached to χ is

$$\tau(\chi) = \sum_{\nu \bmod m} \chi(\nu) \zeta_m^{\nu}.$$

Also, $\overline{\chi}$ denotes the inverse—or equivalently, the complex conjugate—of the character χ. Finally, as customary, $\mu()$, $\varphi()$, and $\zeta()$ denote the Möbius, Euler totient, and Riemann zeta functions, respectively.

Our immediate goal is to represent periodic zeta functions in terms of L-series. It turns out to be easier to do the reverse first. The following result is well known, so we omit the proof.

Lemma 1 *Let m be a positive integer, let χ be a Dirichlet character of modulus m, and let $L(s,\chi)$ be the associated Dirichlet L-series. Then*

$$L(s,\chi) = \frac{1}{m} \sum_{a \bmod m} \chi(a) \sum_{b \bmod m} \zeta_m^{-ab}\, \Phi(s, b/m), \qquad \mathrm{Re}(s) > 1.$$

Lemma 2 *Let a and m be positive integers. Then*

$$\frac{1}{\varphi(m)} \sum_{\chi \in \hat{U}_m} \chi(a)\tau(\overline{\chi})L(s,\chi) = \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m)\, c_m(a-b), \qquad \mathrm{Re}(s) > 1.$$

Proof. First recall that

$$\sum_{\chi \in \hat{U}_m} \overline{\chi}(c)\chi(a) = \begin{cases} \varphi(m) & \text{if } (ac, m) = 1 \text{ and } a \equiv c \bmod m, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that if $(c, m) = 1$, then

$$\frac{\varphi(m)}{m} \sum_{b \bmod m} \zeta_m^{-bc}\, \Phi(s, b/m) = \sum_{\chi \in \hat{U}_m} \chi(c) L(s,\chi); \tag{2}$$

for by Lemma 1,

$$\begin{aligned} \sum_{\chi \in \hat{U}_m} \overline{\chi}(c)\, L(s,\chi) &= \frac{1}{m} \sum_{a \bmod m} \sum_{b \bmod m} \zeta_m^{-ab}\, \Phi(s, b/m) \sum_{\chi \in \hat{U}_m} \overline{\chi}(c)\chi(a) \\ &= \frac{\varphi(m)}{m} \sum_{b \bmod m} \zeta_m^{-bc}\, \Phi(s, b/m). \end{aligned}$$

On the other hand, if $(c, m) > 1$, then clearly

$$\sum_{\chi \in \hat{U}_m} \overline{\chi}(c)\, L(s,\chi) = 0.$$

We now multiply equation (2) by ζ_m^{ac} with $(a, m) = 1$, and then sum over all c modulo m, obtaining

$$\begin{aligned} \frac{\varphi(m)}{m} \sum_{b \bmod m} \sum_{\substack{c \bmod m \\ (c,m)=1}} \zeta_m^{(a-b)c}\, \Phi(s, b/m) &= \sum_{c \bmod m} \zeta_m^{ac} \sum_{\chi \in \hat{U}_m} \overline{\chi}(c) L(s,\chi) \\ &= \sum_{c \bmod m} \zeta_m^{ac} \sum_{\chi \in \hat{U}_m} \overline{\chi}(c) L(s,\chi) \\ &= \sum_{\chi \in \hat{U}_m} \sum_{c \bmod m} \overline{\chi}(c)\zeta_m^{ac}\, L(s,\chi) \\ &= \sum_{\chi \in \hat{U}_m} \chi(a)\tau(\overline{\chi})L(s,\chi). \end{aligned}$$

Rewriting this latter equation in terms of Ramanujan sums completes the proof.
□

We now state the main proposition of this section.

Proposition 1 *Let a and m be coprime positive integers. Then*

$$m^s\Phi(s,a/m) = \sum_{d|m} \frac{d^s}{\varphi(d)} \sum_{\chi\in\hat{U}_d} \chi(a)\tau(\overline{\chi})\, L(s,\chi), \qquad \mathrm{Re}(s)>1.$$

Before proving Proposition 1, we state and prove two lemmata which are used in the proof of Proposition 1.

Lemma 3 *Let $f:\mathbb{Z}\to\mathbb{C}$ be multiplicative and such that for all positive integers m,*

$$F(m) := \sum_{d|m} \mu^2(d) f(d)$$

is non-zero. Furthermore, let

$$g(m) := \sum_{d|m} \frac{\mu(d)}{F(d)}.$$

Then for all positive integers k and m such that k divides m,

$$\sum_{\substack{d|m\\ k|d}} \mu^2(d) f(d) = F(m)\mu^2(k)g(k).$$

In particular,

$$\sum_{\substack{d|m\\ k|d}} \frac{\mu^2(d)}{\varphi(d)} = \frac{m}{\varphi(m)} \frac{\mu^2(k)}{k}.$$

Proof. First, let us define $F(x)=f(x)=0$ if x is not an integer. Next, observe that $F(p^a)=F(p)$ for all positive primes p and positive integers a. We may write m as $\prod_p p^{a_p}$ and k as $\prod_p p^{b_p}$ where p ranges over all positive primes and a_p and b_p are non-negative integers with $b_p \le a_p$. Since F is multiplicative, we have

$$\begin{aligned}
\sum_{\substack{d|m\\ k|d}} \mu^2(d) f(d) &= \prod_{p|m} \left(\sum_{\nu_p=b_p}^{a_p} \mu^2(p^{\nu_p}) f(p^{\nu_p}) \right) \\
&= \prod_{p|m} \left(F(p^{a_p}) - F(p^{b_p-1}) \right) \\
&= \prod_{p|m} F(p^{a_p}) \left(1 - \frac{F(p^{b_p-1})}{F(p^{a_p})} \right) \\
&= F(m) \prod_{p|m} \left(1 - \frac{F(p^{b_p-1})}{F(p^{a_p})} \right). \qquad (3)
\end{aligned}$$

Notice that the final product in (3) vanishes if any $b_p \geq 2$, for then $F(p^{b_p-1}) = F(p) = F(p^{a_p})$. Hence if k is not square-free, then the lemma is trivially true as both sides are equal to 0. Therefore, we may assume henceforth that k is square-free. Now if $b_p = 0$, then $1 - F(p^{b_p-1})/F(p^{a_p}) = 1$, and thus (under the assumption that k is square-free), we may restrict the final product in (3) to primes p for which $b_p = 1$. This yields

$$\begin{aligned}
\sum_{\substack{d|m \\ k|d}} \mu^2(d) f(d) &= F(m) \prod_{p|k} \left(1 - \frac{1}{F(p^{a_p})}\right) = F(m) \prod_{p|k} \left(1 - \frac{1}{F(p)}\right) \\
&= F(m) \sum_{d|k} \frac{\mu(d)}{F(d)} \\
&= F(m) g(k).
\end{aligned}$$

Thus, in general, we have

$$\sum_{\substack{d|m \\ k|d}} \mu^2(d) f(d) = F(m) \mu^2(k) g(k).$$

The special case is obtained by taking $F(m) = m/\varphi(m)$, so that if k is square-free, then

$$g(k) = \sum_{d|k} \frac{\mu(d)\varphi(d)}{d} = \prod_{p|k} \left(1 - \frac{\varphi(p)}{p}\right) = \prod_{p|k} \frac{1}{p} = \frac{1}{k}.$$

This completes the proof of Lemma 1. □

Lemma 4 *Let m be a positive integer and let β be any real number. Then*

$$\sum_{\substack{n=1 \\ (n,m)=1}}^{\infty} \frac{e^{2\pi i \beta n}}{n^s} = \sum_{d|m} \frac{\mu(d)}{d^s} \Phi(s, \beta d).$$

Proof. Let x be any complex number with $|x| \leq 1$. Then

$$\begin{aligned}
\sum_{\substack{n=1 \\ (n,m)=1}}^{\infty} \frac{x^n}{n^s} &= \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{d|(n,m)} \mu(d) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{\substack{d|n \\ d|m}} \mu(d) = \sum_{d|m} \mu(d) \sum_{k=1}^{\infty} \frac{x^{kd}}{(kd)^s} \\
&= \sum_{d|m} \frac{\mu(d)}{d^s} \sum_{k=1}^{\infty} \frac{x^{kd}}{k^s}.
\end{aligned}$$

Replacing x by $e^{2\pi i \beta}$ completes the proof. □

Proof of Proposition 1. First, recall (see eg. [4, p. 238]) that Ramanujan's sum has the explicit representation

$$c_m(k) = \varphi(m) \frac{\mu(m/(m,k))}{\varphi(m/(m,k))}.$$

Hence, we have

$$
\begin{aligned}
&\frac{1}{m}\sum_{b \bmod m} \Phi(s,b/m)\, c_m(a-b) \\
= \;& \frac{1}{m}\sum_{b \bmod m} \Phi(s,b/m)\varphi(m)\frac{\mu(m/(m,a-b))}{\varphi(m/(m,a-b))} \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\sum_{\substack{b \bmod m\\ (a-b,m)=m/d}} \Phi(s,b/m)\frac{\mu(d)}{\varphi(d)} \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\frac{\mu(d)}{\varphi(d)}\sum_{\substack{\nu \bmod d\\ (\nu,d)=1}} \Phi\left(s,\frac{a+m\nu/d}{m}\right) \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\frac{\mu(d)}{\varphi(d)}\sum_{n=1}^{\infty} n^{-s}\sum_{\substack{\nu \bmod d\\ (\nu,d)=1}} \zeta_m^{(a+m\nu/d)n} \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\frac{\mu(d)}{\varphi(d)}\sum_{n=1}^{\infty} n^{-s}\zeta_m^{an}c_d(n) \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\frac{\mu(d)}{\varphi(d)}\sum_{n=1}^{\infty} n^{-s}\zeta_m^{an}\varphi(d)\frac{\mu(d/(n,d))}{\varphi(d/(n,d))} \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\mu(d)\sum_{f|d}\frac{\mu(f)}{\varphi(f)}\sum_{\substack{n=1\\ (n,f)=1}}^{\infty} (nd/f)^{-s}\zeta_{fm/d}^{an} \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\mu(d)\sum_{f|d}\frac{\mu(f)}{\varphi(f)}\left(\frac{d}{f}\right)^{-s}\sum_{\substack{n=1\\ (n,f)=1}}^{\infty} n^{-s}\zeta_{fm/d}^{an}. \qquad (4)
\end{aligned}
$$

But by Lemma 4 the final expression in (4) can be rewritten as

$$\frac{\varphi(m)}{m}\sum_{d|m}\mu(d)\sum_{f|d}\frac{\mu(f)}{\varphi(f)}\left(\frac{d}{f}\right)^{-s}\sum_{\delta|f}\delta^{-s}\mu(\delta)\Phi\left(s,\frac{ad\delta}{fm}\right). \qquad (5)$$

Now transform (5) by changing the variable f to d/f, then letting $k=f\delta$ (noticing that the only non-zero terms occur when d is square-free), then observing that $\sum_{f|k}\varphi(f)=k$, and finally replacing d by kd. Thus, from (4) and (5),

$$
\begin{aligned}
&\frac{1}{m}\sum_{b \bmod m} \Phi(s,b/m)\, c_m(a-b) \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\mu(d)\sum_{f|d}\frac{\mu(d/f)}{\varphi(d/f)}f^{-s}\sum_{\delta|(d/f)}\delta^{-s}\mu(\delta)\Phi(s,af\delta/m) \\
= \;& \frac{\varphi(m)}{m}\sum_{d|m}\sum_{k|d}\frac{\mu^2(d)}{\varphi(d)}k^{-s}\mu(k)\Phi(s,ak/m)\sum_{f|k}\varphi(f)
\end{aligned}
$$

$$= \frac{\varphi(m)}{m} \sum_{k|m} k^{1-s} \mu(k) \Phi(s, ak/m) \sum_{\substack{d|m \\ k|d}} \frac{\mu^2(d)}{\varphi(d)}. \qquad (6)$$

By applying Lemma 3 to (6) and then replacing m/k by d, we find that

$$\begin{aligned} \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m)\, c_m(a-b) &= \sum_{k|m} k^{-s} \mu(k) \Phi(s, ak/m) \\ &= \frac{1}{m^s} \sum_{d|m} d^s \mu(m/d) \Phi(s, a/d). \qquad (7) \end{aligned}$$

Hence by (7) and Lemma 2, we see that

$$\begin{aligned} \frac{1}{m^s} \sum_{d|m} d^s \mu(m/d) \Phi(s, a/d) &= \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m)\, c_m(a-b) \\ &= \frac{1}{\varphi(m)} \sum_{\chi \in \hat{U}_m} \chi(a) \tau(\overline{\chi}) L(s, \chi). \end{aligned}$$

An application of Möbius inversion now completes the proof. □

3 Main Results

We are now in a position to derive our class number formula. To this end, for $|x| \leq 2$ and $2 \leq k \in \mathbb{Z}$, put

$$s(k, x) := \sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{2n}{n} n^k}.$$

Let $0 < \theta < \pi$ and $x = 2\sin\theta/2$. Then [3, p. 61 (2)] $2s(2, x) = \theta^2$ and by formula (2.7) of [5],

$$\begin{aligned} \theta^2 \log(2\sin\theta/2) &= 2\zeta(3) + \sum_{n=1}^{\infty} \frac{(2\sin\theta/2)^{2n}}{\binom{2n}{n} n^3} \\ &- 2\theta \,\mathrm{Im}\, \Phi(2, \theta/2\pi) - 2\mathrm{Re}\, \Phi(3, \theta/2\pi), \qquad (8) \end{aligned}$$

where Re and Im denote the real and imaginary parts of a complex number, respectively. Now substitute $\theta = 2\pi a/m$ with $(a, m) = 1$ and $0 < a < m/2$ in (8) to obtain

$$\begin{aligned} \log\left(2\sin\frac{\pi a}{m}\right) &= \frac{m^2}{2\pi^2 a^2}\zeta(3) + \frac{m^2}{4\pi^2 a^2} \sum_{n=1}^{\infty} \frac{(2\sin\pi a/m)^{2n}}{\binom{2n}{n} n^3} \\ &- \frac{m}{\pi a} \mathrm{Im}\, \Phi\left(2, \frac{a}{m}\right) - \frac{m^2}{2\pi^2 a^2} \mathrm{Re}\, \Phi\left(3, \frac{a}{m}\right). \qquad (9) \end{aligned}$$

In our main result, character sums of consecutive integer powers arise, and it is convenient to fix some notation for these.

Definition 1 *Let m be a positive integer. If χ is a Dirichlet character of modulus m and j is any integer, put*

$$\mathcal{B}_j(\chi) := \sum_{0<a<m/2} a^j \chi(a). \tag{10}$$

We now state and prove our main result.

Theorem 1 *Let m be a positive integer, let ψ be an even primitive character of modulus m, and let $\mathcal{B}_j$ be as in (10). Then*

$$\begin{aligned} L(1,\psi) &= \frac{2\tau(\psi)}{\pi i m^2} \sum_{d|m} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi\in\hat{U}_d \\ \chi\ odd}} \mathcal{B}_{-1}(\chi\overline{\psi})\tau(\overline{\chi})L(2,\chi) \\ &+ \frac{\tau(\psi)}{\pi^2 m^2} \sum_{d|m} \frac{d^3}{\varphi(d)} \sum_{\substack{\chi\in\hat{U}_d \\ \chi\ even}} \mathcal{B}_{-2}(\chi\overline{\psi})\tau(\overline{\chi})\, L(3,\chi) - \frac{m\tau(\psi)}{\pi^2}\mathcal{B}_{-2}(\overline{\psi})\zeta(3) \\ &- \frac{m\tau(\psi)}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^3} \sum_{0<a<m/2} \frac{\overline{\psi}(a)}{a^2} \left(2\sin\frac{\pi a}{m}\right)^{2n}. \end{aligned}$$

Proof. We start with (9) and write $\operatorname{Im}\Phi(2,a/m)$ and $\operatorname{Re}\Phi(3,a/m)$ in terms of L-series via Proposition 1; but first observe that

$$\begin{aligned} &\operatorname{Im}\left(\sum_{\chi \bmod d} \chi(a)\tau(\overline{\chi})L(2,\chi) \right) \\ &= \frac{1}{2i} \sum_{\chi\in\hat{U}_d} \left(\chi(a)\tau(\overline{\chi})L(2,\chi) - \overline{\chi(a)\tau(\overline{\chi})\, L(2,\chi)} \right) \\ &= \frac{1}{2i} \sum_{\chi\in\hat{U}_d} \left(\chi(a)\tau(\overline{\chi})L(2,\chi) - \chi(-1)\overline{\chi}(a)\tau(\chi)L(2,\overline{\chi}) \right), \end{aligned}$$

since $\overline{\tau(\chi)} = \chi(-1)\tau(\overline{\chi})$. Now split the sum over the two terms and in the second sum replace χ by $\overline{\chi}$. The even characters cancel and we obtain

$$\operatorname{Im}\left(\sum_{\chi\in\hat{U}_d} \chi(a)\tau(\overline{\chi})\, L(2,\chi) \right) = \frac{1}{i} \sum_{\substack{\chi\in\hat{U}_d \\ \chi(-1)=-1}} \chi(a)\tau(\overline{\chi})\, L(2,\chi).$$

Similarly, we see that

$$\operatorname{Re}\left(\sum_{\chi\in\hat{U}_d} \chi(a)\tau(\overline{\chi})\, L(3,\chi) \right) = \sum_{\substack{\chi\in\hat{U}_d \\ \chi(-1)=1}} \chi(a)\tau(\overline{\chi})\, L(3,\chi).$$

Thus by (9) and Proposition 1,

$$\begin{aligned}\log\left(2\sin\frac{\pi a}{m}\right) &= \frac{m^2}{2\pi^2a^2}\zeta(3)+\frac{m^2}{4\pi^2a^2}\sum_{n=1}^{\infty}\frac{(2\sin\pi a/m)^{2n}}{\binom{2n}{n}n^3}\\ &- \frac{1}{\pi ima}\sum_{d|m}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi\in\hat{U}_d\\ \chi\,\text{odd}}}\chi(a)\tau(\overline{\chi})\,L(2,\chi)\\ &- \frac{1}{2\pi^2ma^2}\sum_{d|m}\frac{d^3}{\varphi(d)}\sum_{\substack{\chi\in\hat{U}_d\\ \chi\,\text{even}}}\chi(a)\tau(\overline{\chi})\,L(3,\chi). \qquad (11)\end{aligned}$$

Next, recall (see eg. [2, p. 336]) that if ψ is an even primitive character of modulus m, then

$$\begin{aligned}L(1,\psi) &= -\frac{\tau(\psi)}{m}\sum_{a=1}^{m-1}\overline{\psi}(a)\log\left(2\sin\frac{\pi a}{m}\right)\\ &= -\frac{2\tau(\psi)}{m}\sum_{0<a<m/2}\overline{\psi}(a)\log\left(2\sin\frac{\pi a}{m}\right). \qquad (12)\end{aligned}$$

Substituting (12) into (11) completes the proof. □

Let D be a (positive fundamental) discriminant of a real quadratic number field. Let $h(D)$ denote its class number, $\varepsilon=\varepsilon_D$ its fundamental unit >1, and $\chi_D=(D/\cdot)$, the Kronecker symbol, i.e. the Dirichlet character associated with the quadratic field of discriminant D. Then by Dirichlet (see eg. [2, p. 343]), we know that

$$2h(D)\log\varepsilon_D=\sqrt{D}\,L(1,\chi_D).$$

Hence by Theorem 1, using the fact that $\tau(\chi_D)=\sqrt{D}$, we obtain the following class number formula.

Corollary 1 (Class Number Formula)

$$\begin{aligned}h(D)\log\varepsilon_D &= \frac{1}{\pi Di}\sum_{d|D}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi\in\hat{U}_d\\ \chi\ odd}}\mathcal{B}_{-1}(\chi\chi_D)\tau(\overline{\chi})\,L(2,\chi)\\ &+ \frac{1}{2\pi^2D}\sum_{d|D}\frac{d^3}{\varphi(d)}\sum_{\substack{\chi\in\hat{U}_d\\ \chi\ even}}\mathcal{B}_{-2}(\chi\chi_D)\tau(\overline{\chi})\,L(3,\chi)\\ &- \frac{D^2}{2\pi^2}\mathcal{B}_{-2}(\chi_D)\zeta(3)\\ &- \frac{D^2}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}n^3}\sum_{0<a<D/2}\frac{\chi_D(a)}{a^2}\left(2\sin\frac{\pi a}{D}\right)^{2n}.\end{aligned}$$

3.1 A Computation

As an amusing conclusion, we now show how to use our class number formula to compute $h(5)$, the class number of the quadratic field $\mathbb{Q}(\sqrt{5})$. Since the discriminant $D = 5$, the only relevant moduli of characters are $m = 1$ and $m = 5$. For $m = 1$, the unique character is the even constant character 1. For $m = 5$, we have four characters determined by the homomorphisms from $(\mathbb{Z}/5\mathbb{Z})^\times$ into $\mathbb{C}^\times$, namely χ_ν for $\nu = 0, 1, 2, 3$ determined by $\chi_\nu(2) = i^\nu$. Notice that $\overline{\chi_1} = \chi_3$ and that $\chi_2 = (5/\cdot) = \chi_5$, the Kronecker character modulo 5.

By Corollary 1, we have $h(5) = (A + B + C + S)/\log \varepsilon_5$, where

$$\begin{aligned}
A &= \frac{5}{4\pi i}\left(\mathcal{B}_{-1}(\chi_3)\tau(\chi_3)L(2,\chi_1) + \mathcal{B}_{-1}(\chi_1)\tau(\chi_1)L(2,\chi_3)\right), \\
B &= \frac{1}{10\pi^2}\Bigg(\mathcal{B}_{-2}(\chi_5)\tau(1)\zeta(3) \\
&\qquad + \frac{125}{4}\left(\mathcal{B}_{-2}(\chi_5)\tau(\chi_0)L(3,\chi_0) + \mathcal{B}_{-2}(\chi_0)\tau(\chi_5)L(3,\chi_5)\right)\Bigg) \\
C &= -\frac{25}{2\pi^2}\mathcal{B}_{-2}(\chi_5)\zeta(3) \\
S &= -\frac{25}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}n^3}\left(\chi_5(1)(2\sin\pi/5)^{2n} + \tfrac{1}{4}\chi_5(2)(2\sin 2\pi/5)^{2n}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_{-1}(\chi_1) &= \chi_1(1) + \tfrac{1}{2}\chi_1(2) = 1 + \tfrac{1}{2}i \\
\mathcal{B}_{-1}(\chi_3) &= 1 - \tfrac{1}{2}i \\
\mathcal{B}_{-2}(\chi_0) &= 1 + \tfrac{1}{4} = \tfrac{5}{4} \\
\mathcal{B}_{-2}(\chi_5) &= 1 - \tfrac{1}{4} = \tfrac{3}{4} \\
\tau(1) &= 1 \\
\tau(\chi_0) &= -1 \\
\tau(\chi_5) &= \sqrt{5} \\
\tau(\chi_1) &= \zeta_5 + i\zeta_5^2 - i\zeta_5^3 - \zeta_5^4 = \left(i + \frac{1-\sqrt{5}}{2}\right)\sqrt{\frac{5+\sqrt{5}}{2}} \\
\tau(\chi_3) &= \zeta_5 - i\zeta_5^2 + i\zeta_5^3 - \zeta_5^4 = \left(i + \frac{\sqrt{5}-1}{2}\right)\sqrt{\frac{5+\sqrt{5}}{2}} \\
L(3,\chi_0) &= \left(1 - 5^{-3}\right)\zeta(3) = \frac{124}{125}\zeta(3).
\end{aligned}$$

In order to evaluate $L(s,\chi_\nu)$ for $\nu = 0, 1, 2, 3$ and $s = 2, 3$ we write

$$L(s,\chi_\nu) = 5^{-s}\sum_{r=1}^{4}\chi_\nu(r)\zeta(s, r/5),$$

where $\zeta(s,a)=\sum_{n=0}^{\infty}(n+a)^{-s}$ is the Hurwitz zeta function. Hence to evaluate these L-series, it suffices to evaluate the Hurwitz zeta functions. The following table gives the appropriate approximations.

r	$\zeta(2,r/5)$	$\zeta(3,r/5)$
1	26.26737720...	125.73901805...
2	7.27535659...	16.1195643...
3	3.63620967...	4.98141576...
4	2.29947413...	2.21505785...

Thus, we find that

$$\begin{aligned} L(2,\chi_1) &= 0.95871612... + (0.14556587...)i \\ L(2,\chi_3) &= 0.95871612... - (0.14556587...)i \\ L(3,\chi_5) &= 0.85482476... \end{aligned}$$

Furthermore, $\zeta(3)=1.20205690...$, so that

$$\begin{aligned} A &= 1.24907310... \\ B &= 0.48248793... \\ C &= -1.14181713... \\ S &= -0.10853146... \end{aligned}$$

Finally, the fundamental unit

$$\varepsilon_5 = \frac{1+\sqrt{5}}{2},$$

(which generally can be computed efficiently by continued fractions).

From all of this we obtain $h(5) = 1.0000000... + (0.0000000...)i$, whence $h(5)=1$.

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