REMARK ON CAVALIERI'S QUADRATURE FORMULA

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With regard to Cavalieri's quadrature formula [1] for $0 < a < b$ and integer n:

$$\int_a^b x^n\,dx = \begin{cases} \frac{b^{n+1}-a^{n+1}}{n+1}, & \text{if } n \neq -1, \\ \log\frac{b}{a}, & \text{if } n = -1, \end{cases} \tag{1}$$

there is an amusing sense in which the exceptional case contains the others. For example, if $|t| < 1$ then

$$\sum_{n=0}^{\infty} t^n \int_0^1 x^n\,dx = \int_0^1 \sum_{n=0}^{\infty} (tx)^n = \int_0^1 \frac{dx}{1-tx} = t^{-1}\log(1-t)^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{n+1}.$$

More generally, if $0 < a < b$ and $|t| < \min(a, 1/b)$ then

$$\sum_{n=0}^{\infty} t^n \int_a^b x^n\,dx = t^{-1}\log\left(\frac{1-at}{1-bt}\right), \qquad \sum_{n=1}^{\infty} t^n \int_a^b x^{-n}\,dx = t\log\left(\frac{a-t}{b-t}\right).$$

The quadrature formula (1) follows on expanding the logarithms in powers of t and comparing coefficients.

References

[1] N. J. Wildberger, A new proof of Cavalier's quadrature formula, this Monthly **109** (2002), 843–845.

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