

# MORE APÉRY-LIKE FORMULAE: ON REPRESENTING VALUES OF THE RIEMANN ZETA FUNCTION BY INFINITE SERIES DAMPED BY CENTRAL BINOMIAL COEFFICIENTS

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ABSTRACT. We prove an identity for a certain infinite series with two free parameters. The presence of central binomial coefficients improves the rate of convergence of the original series; hence the identity can be considered as a series acceleration formula. By viewing the two sides of the identity as generating functions and specializing the parameters appropriately, we obtain series acceleration formulae for values of the Riemann zeta function. Special cases of our results include the classical formula for  $\zeta(3)$  popularized by Apéry, the formula of Kocher and Leshchiner for  $\zeta(2n+3)$ , the more compact formula of Borwein and Bradley for  $\zeta(4n+3)$ , and others. Of our main infinite series identity, we give some equivalent reformulations, including some interesting finite identities and strange hypergeometric series evaluations. By setting the free parameter equal to zero in these latter identities, we obtain identities that were previously conjectured by Borwein, Bradley, and Chu, and that were subsequently proved by Almkvist and Granville.

## 1. INTRODUCTION

The Riemann zeta function is defined for complex  $s$  with  $\Re(s) > 1$  by its Dirichlet series representation

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

We are interested in series acceleration formulae for special values, of which the following are typical:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}, \tag{1.1}$$

$$\zeta(5) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2}, \tag{1.2}$$

$$\zeta(7) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} + \frac{25}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4}. \tag{1.3}$$

Due to the role it played in Apéry's proof [2] of the irrationality of  $\zeta(3)$ , equation (1.1) is now quite widely known, but it can be traced back to Hjortnaes [6]. The author first encountered (1.2) while perusing Koecher's charming book [8],

recurrence?

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and was subsequently inspired [3] to search for more such identities by computing the relevant series to high precision and then using a linear relations finding algorithm to reveal the rational constant factors needed to make the identity true. The formula (1.3) was the first of many that were found using this approach.

One way to gain a better understanding of formulae such as (1.1)–(1.3) is to view them as the result of comparing corresponding coefficients in an identity satisfied by appropriate generating functions. Let  $z$  be a complex number with  $|z| < 1$ , and let  $n$  be a non-negative integer. Comparing coefficients of  $z^{2n}$  in [7]

zero as  $z \rightarrow \infty$

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-z^2/k^2)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \left( \frac{1}{2} + \frac{2}{1-z^2/k^2} \right) \prod_{j=1}^{k-1} (1-z^2/j^2) \quad (1.4)$$

yields a formula (cf. also [10]) for  $\zeta(2n+3)$ , of which (1.1) and (1.2) correspond to the cases  $n=0$  and  $n=1$ , respectively. Similarly, comparing coefficients of  $z^{4n}$  in [3]

zero as  $z \rightarrow \infty$

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-z^4/k^4)} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \frac{1}{1-z^4/k^4} \prod_{j=1}^{k-1} \frac{1+4z^4/j^4}{1-z^4/j^4} \quad (1.5)$$

yields a formula for  $\zeta(4n+3)$ , of which (1.1) and (1.3) correspond to the cases  $n=0$  and  $n=1$ , respectively. Interestingly, the formula for  $\zeta(7)$  arising from the  $n=2$  case of (1.2) is quite distinct from (1.3), in that the latter apparently contains no redundant terms, in contrast to the former. For details, the interested reader is referred to the discussion concerning formula redundancy in [3] or [4].

The generating function identities (1.4) and (1.5) are actually valid for all complex  $z$  with the exception of the obvious poles. Herein, we prove the following conjecture of Henri Cohen, which simultaneously generalizes (1.4) and (1.5).

**Theorem 1.** *Let  $b$  and  $c$  be complex numbers with neither  $(-b + \sqrt{b^2 - 4c})/2$  nor  $(-b - \sqrt{b^2 - 4c})/2$  equal to the square of a positive integer. Then*

$$\sum_{k=1}^{\infty} \frac{k}{k^4 + bk^2 + c} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k \binom{2k}{k}} \frac{5k^2 + b}{k^4 + bk^2 + c} \prod_{j=1}^{k-1} \frac{j^4 + 2bj^2 + b^2 - 4c}{j^4 + bj^2 + c}.$$

Setting  $b = -z^2$  and  $c = 0$  in Theorem 1 yields (1.4); setting  $b = 0$  and  $c = -z^4$  yields (1.5). Additional consequences of Theorem 1 are stated below.

Gosper reflect with  $c$ .

Show both sides tend to zero as either  $b \rightarrow \infty$  or  $c \rightarrow \infty$ .

Do asymptotic expansion in reciprocal powers of  $b$  and  $c$ .

**Corollary 1.** *Let  $z$  be a complex number not equal to a non-zero integer. Then*

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-z^2/k^2)^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \frac{1}{1-z^2/k^2} \left( 1 + \frac{3/2}{1-z^2/k^2} \right) \prod_{j=1}^{k-1} \frac{1-4z^2/j^2}{(1-z^2/j^2)^2}.$$

*Proof.* Set  $b = -2z^2$ ,  $c = z^4$  in Theorem 1, and simplify.  $\square$

Less trivially, we have the following evaluations for generalized hypergeometric series.

**Corollary 2.** *Let  $n$  be a positive integer, and let  $c$  be a complex number. Then with the exception of the poles at  $c = j^2n^2$ ,  $n + 1 \leq j \leq 2n$ , we have*

$$\begin{aligned} {}_7F_6 \left( \begin{matrix} n, n+1 \pm \sqrt{(n^4+c)/5n^2}, 2n \pm \sqrt{c}/n, \pm\sqrt{c}/n \\ n+1/2, n \pm \sqrt{(n^4+c)/5n^2}, 2n+1, n+1 \pm \sqrt{c}/n \end{matrix} \middle| -\frac{1}{4} \right) \\ = \prod_{j=n+1}^{2n} \left( 1 - \frac{c}{j^2n^2} \right)^{-1}. \end{aligned}$$

**Corollary 3.** *Let  $n$  be a positive integer, and let  $b$  be a complex number. Then, with the exception of the poles at  $b = -n^2 - j^2$ ,  $n + 1 \leq j \leq 2n$ , we have*

$$\begin{aligned} {}_7F_6 \left( \begin{matrix} n, n+1 \pm i\sqrt{b/5}, n \pm \sqrt{-b \pm 2in\sqrt{n^2+b}} \\ n+1/2, 2n+1, n \pm i\sqrt{b/5}, n+1 \pm i\sqrt{n^2+b} \end{matrix} \middle| -\frac{1}{4} \right) \\ = \prod_{j=n+1}^{2n} \left( 1 + \frac{n^2+b}{j^2} \right)^{-1}. \end{aligned}$$

Here and throughout,  $i^2 = -1$ , and all possible choices of signs are to be taken wherever sign ambiguities are present. Setting  $b = 0$  in Corollary 3 yields an equivalent version of Corollary 2.3 of [3]. We'll see that Corollary 3 follows from Theorem 1 as a consequence of comparing residues at the poles on each side of the identity. By appealing to Mittag-Leffler's theorem, we can actually show that Corollary 3 implies Theorem 1, and hence the two results are equivalent in that they readily imply each other.

The expressions on the right hand side of Corollaries 3 and 2 suggest that these formulas might be proved by verifying them for only  $n + 1$  distinct values of the parameter. For example, it is clear that both sides of Corollary 2 are rational functions of  $c$ , with denominator of degree  $n$  in each case. If it could be shown that the numerator on the left has degree zero in  $c$  as it does on the right, then it would suffice to check the result for  $n + 1$  distinct values of  $c$ . The values  $c = r^2n^2$  with  $r = 0, 1, 2, \dots, n$  cause the hypergeometric series in Corollary 2 to terminate in a finite sum. If this substitution is made and the factors rearranged, one obtains the following result.

**Corollary 4.** *Let  $n$  be a positive integer, and let  $r$  be an integer satisfying  $0 \leq r \leq n$ . Then*

$$\begin{aligned} \sum_{k=0}^r \frac{(-1)^k}{n+k} \frac{n(4n^2+10nk+5k^2-r^2)}{(4n^2-r^2)\binom{2n+2k}{n+k}} \prod_{j=n}^{n+k-1} \frac{((j+n)^2-r^2)((j-n)^2-r^2)}{((j+1)^2-n^2)((j+1)^2-r^2)} \\ = \frac{\binom{4n}{2n-r}}{\binom{4n}{2n}\binom{2n}{n-r}}. \end{aligned}$$

Theorem 1 is proved by reducing the problem to that of proving the following two equivalent finite identities, which are easily seen to be the respective  $b$ -analogues of Lemmata 4.1 and 5.2 of [3].

Lemmas?

**Lemma 2.** *Let  $n$  be a positive integer, and let  $b$  be a complex number. Then*

$$\sum_{k=1}^n \binom{2k}{k} \frac{(n^2+b)(5k^2+b)}{4n^4+4bn^2+k^4+2bk^2+b^2} \prod_{j=1}^{k-1} \frac{n^4+bn^2-j^4-bj^2}{4n^4+4bn^2+j^4+2bj^2+b^2} = 2.$$

**Lemma 3.** *Let  $n$  be a positive integer, and let  $b$  be a complex number. Then*

$$2 \sum_{k=1}^n \frac{n^2 + b}{k^2 + b} \prod_{j=1}^{n-1} (4k^4 + 4bk^2 + j^4 + 2bj^2 + b^4) \Big/ \prod_{\substack{j=1 \\ j \neq k}}^n (k^4 + bk^2 - j^4 - bj^2) = \binom{2n}{n}.$$

## 2. REDUCTION TO A FINITE IDENTITY

By partial fractions, if  $k$  is a positive integer, then

$$\frac{1}{k^4 + bk^2 + c} \prod_{j=1}^{k-1} \frac{j^4 + 2bj^2 + b^2 - 4c}{j^4 + bj^2 + c} = \sum_{j=1}^k \frac{c_j(k)}{j^4 + bj^2 + c},$$

where for integers  $1 \leq n \leq k$ ,

$$\begin{aligned} c_n(k) &= \prod_{j=1}^{k-1} (j^4 + 2bj^2 + b^2 + 4n^4 + 4bn^2) \Big/ \prod_{\substack{j=1 \\ j \neq n}}^k (j^4 + bj^2 - n^4 - bn^2) \\ &= \prod_{j=1}^{k-1} (n^2 + (n-j)^2 + b)(n^2 + (n+j)^2 + b) \Big/ \prod_{\substack{j=1 \\ j \neq n}}^k (j^2 - n^2)(n^2 + j^2 + b) \\ &= \frac{(-1)^{n+1} 2n^2}{(k-n)!(k+n)!} \prod_{j=0}^{k-n-1} \prod_{j=k+1}^{k+n-1} (n^2 + j^2 + b). \end{aligned} \quad (2.1)$$

For integers  $1 \leq n \leq k$ , let

$$t_n(k) = \frac{(-1)^{k+1} (5k^2 + b)}{2k \binom{2k}{k}} c_n(k). \quad (2.2)$$

We view both sides of Theorem 1 as meromorphic functions of  $c$ . By Mittag-Leffler's theorem, it suffices to prove that for all positive integers  $n$ ,

$$\sum_{k \geq n} t_n(k) = n \quad (2.3)$$

We'd like to extend the definition of  $t_n(k)$  to include integer values of  $k$  less than  $n$ . To this end, we temporarily assume that  $b > 0$  and define, first for positive integers  $1 \leq n \leq k$ ,

$$\begin{aligned} \alpha_n(k) &= \frac{t_n(k)}{t_n(k+1)} \\ &= \frac{2(5k^2 + b)(2k+1)(n^2 + (k+1)^2 + b)(n+k+1)(n-k-1)}{k(5(k+1)^2 + b)(n^2 + (k+n)^2 + b)(n^2 + (k-n)^2 + b)}. \end{aligned} \quad (2.4)$$

For other values of  $k \neq 0$ , define  $\alpha_n(k)$  by the rational function of  $k$  given explicitly in (2.4). Then  $\alpha_n(n-1) = 0$ , and for integers  $1 \leq k < n$ , we define

$$t_n(k) = t_n(n) \prod_{j=k}^{n-1} \alpha_n(j) = 0,$$

which is consistent with what we would obtain if we rewrote  $t_n(k)$  in terms of gamma functions via (2.1) and (2.2). Thus, in view of (2.3), it suffices to prove

elaborate

that for all positive integers  $n$ ,

$$\sum_{k=1}^{\infty} t_n(k) = n. \quad (2.5)$$

We shall need to evaluate  $t_n(k)$  when  $k$  is a non-positive integer also. Since  $\alpha_n(0)$  is undefined, we directly compute from (2.1) and (2.2)

$$\begin{aligned} t_n(0) &= \lim_{k \rightarrow 0} \frac{(-1)^n b n^2}{k \Gamma(k-n+1) \Gamma(k+n+1)} \prod_{j=0}^{-n-1} \prod_{j=1}^{n-1} (n^2 + j^2 + b) \\ &= -bn \prod_{j=1}^{n-1} (n^2 + j^2 + b) \Big/ \prod_{j=1}^n (n^2 + j^2 + b) \\ &= -\frac{bn}{2n^2 + b}, \end{aligned}$$

and then set for positive integer  $k$ ,

$$\begin{aligned} t_n(-k) &= t_n(0) \prod_{j=1}^k \alpha_n(-j) \\ &= -\frac{bn}{2n^2 + b} \prod_{j=1}^k \frac{2(5j^2 + b)(1-2j)(n^2 + (1-j)^2 + b)(n^2 - (1-j)^2)}{(5(1-j)^2 + b)(-j)(n^2 + (n+j)^2 + b)(n^2 + (n-j)^2 + b)} \\ &= -\frac{n(5k^2 + b)}{2n^2 + b} \prod_{j=1}^k \frac{2j(2j-1)}{j^2} \frac{(n^2 + (j-1)^2 + b)(n^2 - (j-1)^2)}{n^2 + (n+j)^2 + b)(n^2 + (n-j)^2 + b)} \\ &= -\frac{n(5k^2 + b)}{2n^2 + b} \binom{2k}{k} \prod_{j=1}^k \frac{(n^2 + (j-1)^2 + b)(n^2 - (j-1)^2)}{4n^4 + 4bn^2 + j^4 + 2bj^2 + b^2}. \end{aligned} \quad (2.6)$$

Define

$$T_n(m) = \sum_{k=-\infty}^m t_n(k) = \sum_{k=-n}^m t_n(k), \quad m \geq -n.$$

We claim that  $\lim_{m \rightarrow \infty} T_n(m) = 0$ . Then (2.5) is equivalent to  $T_n(0) = -n$ . In other words, if  $n$  is a positive integer, then we need to prove that

$$-n = \sum_{k=0}^n t_n(-k) = -\sum_{k=0}^n \frac{n(5k^2 + b)}{2n^2 + b} \binom{2k}{k} \prod_{j=1}^k \frac{(n^2 + (j-1)^2 + b)(n^2 - (j-1)^2)}{4n^4 + 4bn^2 + j^4 + 2bj^2 + b^2}.$$

Equivalently,

$$\begin{aligned} &2n^2 + b \\ &= b + \sum_{k=1}^n \binom{2k}{k} \frac{(5k^2 + b)(n^2 + b)n^2}{4n^4 + 4bn^2 + k^4 + 2bk^2 + b^2} \prod_{j=1}^{k-1} \frac{(n^2 + j^2 + b)(n^2 - j^2)}{4n^4 + 4bn^2 + j^4 + 2bj^2 + b^2}, \end{aligned}$$

which, after cancelling  $b$  and dividing both sides by  $n^2$ , is easily seen to be equivalent to Lemma 2.

## 3. PROOF OF LEMMATA 2 AND 3

neither has this

We first show that Lemmata 2 and 3 imply each other. This fact is a special case of the inverse pair relationship [5, 9]

$$f(n) = \sum_{k=r}^n \frac{a_n d_n + b_n c_n}{d_k} \frac{\varphi(c_k/d_k; n)}{\psi_k(-c_k/d_k; n+1)} g(k)$$

if and only if

$$g(n) = \sum_{k=r}^n \frac{\psi(-c_n/d_n; k)}{\varphi(c_n/d_n; k+1)} f(k),$$

where

$$\varphi(x; k) := \prod_{j=0}^{k-1} (a_j + x b_j), \quad \psi(x; k) := \prod_{j=0}^{k-1} (c_j + x d_j), \quad \psi_m(x; k) := \prod_{\substack{j=0 \\ j \neq m}}^{k-1} (c_j + x d_j).$$

Setting  $r = 1$ ,  $a_j = (j^2 + b)^2$ ,  $b_j = 4$ ,  $c_j = j^4 + b j^2$ ,  $d_j = 1$ ,  $f(k) = (-1)^k (5k^2 + b) \binom{2k}{k}$ , and  $g(n) = 2n^2 / (2n^2 + b)^2$  yields the claimed inverse pair relationship between Lemmata 2 and 3. Thus, it suffices to prove Lemma 3.

*Proof of Lemma 3.* Let  $S_n(b)$  denote the sum in Lemma 3, and factor the expressions that occur in the products. Then

$$S_n(b) = 2 \sum_{k=1}^n \left( \frac{n^2 + b}{k^2 + b} \right) \frac{\prod_{j=1}^{n-1} (k^2 + (j-k)^2 + b)(k^2 + (j+k)^2 + b)}{\prod_{\substack{j=1 \\ j \neq k}}^n (k^2 + j^2 + b)(k^2 - j^2)}. \quad (3.1)$$

But for  $1 \leq k \leq n$ ,

$$\prod_{\substack{j=1 \\ j \neq k}}^n (k^2 - j^2)^{-1} = \frac{(-1)^{n-k} 2k^2}{(n-k)!(n+k)!} = \frac{(-1)^{n-k} 2k^2}{(2n)!} \binom{2n}{n-k}, \quad (3.2)$$

and by re-indexing the products in the numerator of (3.1), we find that

$$\begin{aligned} & \prod_{j=1}^{n-1} (k^2 + (j-k)^2 + b)(k^2 + (j+k)^2 + b) \Big/ \prod_{\substack{j=1 \\ j \neq k}}^n (k^2 + j^2 + b) \\ &= \prod_{\substack{0 \leq j \leq k-1 \\ 1 \leq j \leq n-k-1}} (k^2 + j^2 + b) \prod_{k+1 \leq j \leq n+k-1} (k^2 + j^2 + b) \Big/ \prod_{\substack{1 \leq j \leq n \\ j \neq k}} (k^2 + j^2 + b) \\ &= \prod_{\substack{0 \leq j < n-k \\ n < j < n+k}} (k^2 + j^2 + b). \end{aligned} \quad (3.3)$$

After substituting (3.2) and (3.3) into (3.1), there comes

$$S_n(b) = \frac{4}{(2n)!} \sum_{k=1}^n (-1)^{n-k} \binom{2n}{n-k} k^2 \binom{n^2+b}{k^2+b} \prod_{\substack{0 \leq j < n-k \\ n < j < n+k}} (k^2 + j^2 + b). \quad (3.4)$$

For positive integers  $n$  and  $k$  with  $1 \leq k \leq n$  and complex  $z$ , define

$$f_{n,k}(z) = \frac{4}{(2n)!} \prod_{\substack{0 \leq j < n-k \\ n < j < n+k}} (z + j^2).$$

By Proposition 7 of [1], there exist polynomials  $p_r = p_{n,r}$  of degree at most  $r$  such that

$$f_{n,k}(z) = \sum_{r=0}^{n-1} p_r(k^2) z^{n-1-r}.$$

We claim that

$$p_{n-1}(k^2) = \begin{cases} 0, & \text{if } 1 \leq k \leq n-1; \\ \frac{1}{n^2} \binom{2n}{n}, & \text{if } k = n. \end{cases} \quad (3.5)$$

To see the claim, observe that  $p_{n-1}(k^2)$  is simply the constant term with respect to the variable  $z$  in  $f_{n,k}(z)$ , and is therefore equal to

$$f_{n,k}(0) = \frac{4}{(2n)!} \prod_{\substack{0 \leq j < n-k \\ n < j < n+k}} j^2.$$

If  $k < n$ , this product vanishes due to the presence of the  $j = 0$  term. If  $k = n$ , we get

$$p_{n-1}(n^2) = f_{n,n}(0) = \frac{4}{(2n)!} \prod_{n < j < 2n} j^2 = \frac{1}{n^2 (2n)!} \prod_{n < j \leq 2n} j^2 = \frac{1}{n^2 (2n)!} \left( \frac{(2n)!}{n!} \right)^2,$$

as stated.

It follows that

$$\begin{aligned} S_n(b) &= \sum_{k=1}^n (-1)^{n-k} \binom{2n}{n-k} k^2 \binom{n^2+b}{k^2+b} \sum_{r=0}^{n-1} p_r(k^2) (k^2 + b)^{n-1-r} \\ &= \sum_{r=0}^{n-1} \sum_{k=1}^n (-1)^{n-k} \binom{2n}{n-k} k^2 q_r(k^2), \end{aligned}$$

where

$$q_r(k^2) := \binom{n^2+b}{k^2+b} (k^2 + b)^{n-1-r} p_r(k^2)$$

is a polynomial in  $k^2$  of degree at most  $n-2$  if  $r$  is a non-negative integer strictly less than  $n-1$ . Linearity and the identity [1, Lemma 6]

$$\sum_{k=1}^n (-1)^{n-k} \binom{2n}{n-k} k^{2m} = 0, \quad \text{for integers } 1 \leq m < n,$$

in conjunction with (3.5) now imply that

$$S_n(b) = \sum_{k=1}^n (-1)^{n-k} \binom{2n}{n-k} k^2 \binom{n^2+b}{k^2+b} p_{n-1}(k^2) = n^2 p_{n-1}(n^2) = \binom{2n}{n},$$

as required.  $\square$

#### 4. HYPERGEOMETRIC REFORMULATIONS

As usual, if  $r$  is a non-negative integer and  $a$  is complex, then the rising factorial with  $r$  consecutive factors is denoted by

$$(a)_r = \prod_{j=0}^{r-1} (a+j), \quad (a)_0 := 1,$$

and for non-negative integers  $p$  and  $q$ , the generalized hypergeometric series

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) := \sum_{r=0}^{\infty} \frac{z^r}{r!} \frac{\prod_{j=1}^p (a_j)_r}{\prod_{j=1}^q (b_j)_r}$$

is defined at least for complex  $z$  with  $|z| < 1$  if the parameters  $a_j$  and  $b_j$  are complex numbers with no  $b_j$  equal to zero or a negative integer. In this notation of generalized hypergeometric series, Theorem 1 states that

$$\begin{aligned} \left( \frac{5+b}{1+b+c} \right) {}_8F_7 \left( \begin{matrix} 1, 1, 2 \pm i\sqrt{b/5}, 1 \pm \sqrt{-b \pm 2\sqrt{c}} \\ \frac{3}{2}, 1 \pm i\sqrt{b/5}, 2 \pm \sqrt{-\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2-4c}} \end{matrix} \middle| -\frac{1}{4} \right) \\ = \sum_{k=1}^{\infty} \frac{4k}{k^4 + bk^2 + c}. \end{aligned} \quad (4.1)$$

Assuming the truth of Theorem 1 in the form (4.1), we now deduce, as a consequence, Corollary 3.

*Theorem 1 implies Corollary 3.* Replace  $c$  by  $-z$  in (4.1) and view both sides as meromorphic functions of  $z$  with simple poles at  $z = n^4 + bn^2$ , where  $n$  is a positive integer. We shall see that Corollary 3 follows by equating the corresponding residues. Since the residue at the pole  $z = n^4 + bn^2$  on the right hand side of (4.1)



is clearly equal to  $-4n$ , we must have

$$\begin{aligned}
-4n &= \lim_{z \rightarrow n^4 + bn^2} \frac{(z - n^4 - bn^2)(5 + b)}{1 + b - z} \\
&\quad \times \sum_{k \geq n-1} \frac{(1)_k (1)_k (2 \pm i\sqrt{b/5})_k (1 \pm \sqrt{-b \pm 2i\sqrt{z}})_k}{\left(\frac{3}{2}\right)_k (1 \pm i\sqrt{b/5})_k (2 \pm \sqrt{-\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 + 4z}})_k (-4)^k k!} \\
&= -\frac{(5 + b)(1)_{n-1} (2 \pm i\sqrt{b/5})_{n-1} (1 \pm \sqrt{-b \pm 2in\sqrt{n^2 + b}})_{n-1}}{\left(\frac{3}{2}\right)_{n-1} (1 \pm i\sqrt{b/5})_{n-1} (-4)^{n-1} \prod_{j=1}^{n-1} (j^4 + bj^2 - n^4 - bn^2)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(n)_k (n + 1 \pm i\sqrt{b/5})_k (n \pm \sqrt{-b \pm 2in\sqrt{n^2 + b}})_k}{(n + 1/2)_k (n \pm i\sqrt{b/5})_k (-4)^k \prod_{j=n+1}^{n+k} (j^4 + bj^2 - n^4 - bn^2)} \\
&= -\frac{(5 + b)(n - 1)! (n \pm i\sqrt{b/5})}{\left(\frac{3}{2}\right)_{n-1} (1 \pm i\sqrt{b/5}) 4^{n-1}} \prod_{j=1}^{n-1} \frac{4n^4 + 4bn^2 + j^4 + 2bj^2 + b^2}{n^4 + bn^2 - j^4 - bj^2} \\
&\quad \times {}_7F_6 \left( \begin{matrix} n, n + 1 \pm i\sqrt{b/5}, n \pm \sqrt{-b \pm 2in\sqrt{n^2 + b}} \\ n + 1/2, n \pm i\sqrt{b/5}, 2n + 1, n + 1 \pm i\sqrt{n^2 + b} \end{matrix} \middle| -\frac{1}{4} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&{}_7F_6 \left( \begin{matrix} n, n + 1 \pm i\sqrt{b/5}, n \pm \sqrt{-b \pm 2in\sqrt{n^2 + b}} \\ n + 1/2, n \pm i\sqrt{b/5}, 2n + 1, n + 1 \pm i\sqrt{n^2 + b} \end{matrix} \middle| -\frac{1}{4} \right) \\
&= \frac{2n^2}{5n^2 + b} \binom{2n}{n} \prod_{j=1}^{n-1} \frac{n^4 + bn^2 - j^4 - bj^2}{4n^4 + 4bn^2 + j^4 + 2bj^2 + b^2}. \quad (4.2)
\end{aligned}$$

We remark in passing that setting  $b = 0$  in (4.2) yields Corollary 2.3 of [3]. By factoring the terms in the product on the right hand side of (4.2) and re-indexing as necessary, we find that

$$\begin{aligned}
\prod_{j=1}^{n-1} \frac{n^4 + bn^2 - j^4 - bj^2}{4n^4 + 4bn^2 + j^4 + 2bj^2 + b^2} &= \prod_{j=1}^{n-1} \frac{(n^2 - j^2)(n^2 + j^2 + b)}{(n^2 + (n - j)^2 + b)(n^2 + (n + j)^2 + b)} \\
&= \prod_{j=1}^{n-1} \frac{(n - j)(n + j)}{n^2 + (n + j)^2 + b} \\
&= \frac{(2n)!}{2n^2} \prod_{j=n+1}^{2n-1} (n^2 + j^2 + b)^{-1},
\end{aligned}$$

which when substituted back into (4.2) yields

$$\begin{aligned}
&{}_7F_6 \left( \begin{matrix} n, n + 1 \pm i\sqrt{b/5}, n \pm \sqrt{-b \pm 2in\sqrt{n^2 + b}} \\ n + 1/2, n \pm i\sqrt{b/5}, 2n + 1, n + 1 \pm i\sqrt{n^2 + b} \end{matrix} \middle| -\frac{1}{4} \right) \\
&= \frac{(2n)!}{5n^2 + b} \binom{2n}{n} \prod_{j=n+1}^{2n-1} (n^2 + j^2 + b)^{-1} = \prod_{j=n+1}^{2n} \frac{j^2}{n^2 + j^2 + b},
\end{aligned}$$

as claimed.  $\square$

Thus, we have shown that Corollary 3 follows from Theorem 1. By applying Mittag-Leffler's theorem, we next show that Corollary 3 implies Theorem 1.

*Theorem 1 implies Corollary 2.* Replace  $b$  by  $-z$  in (4.1) and view both sides as meromorphic functions of  $z$  with simple poles at  $z = n^2 + c/n^2$ , where  $n$  is a positive integer. We shall see that Corollary 2 follows by equating the corresponding residues. Since the residue at the pole  $z = n^2 + c/n^2$  on the right hand side of (4.1) is clearly equal to  $-4/n$ , we must have

$$\begin{aligned}
-\frac{4}{n} &= \lim_{z \rightarrow n^2 + c/n^2} \frac{(z - n^2 - c/n^2)(5 - z)}{1 + c - z} \\
&\quad \times \sum_{k \geq n-1} \frac{(1)_k (1)_k (2 \pm \sqrt{z/5})_k (1 \pm \sqrt{z \pm 2\sqrt{c}})_k}{(3/2)_k (1 \pm \sqrt{z/5})_k (-4)^k k! \prod_{j=2}^{k+1} (j^4 - j^2 z + c)} \\
&= \lim_{z \rightarrow n^2 + c/n^2} \frac{(z - n^2 - c/n^2)(5 - z) (1)_{n-1} (2 \pm \sqrt{z/5})_{n-1} (1 \pm \sqrt{z \pm 2\sqrt{c}})_{n-1}}{(3/2)_{n-1} (1 \pm \sqrt{z/5})_{n-1} (-4)^{n-1} \prod_{j=1}^n (j^4 - zj^2 + c)} \\
&\quad \times \sum_{k \geq 0} \frac{(n)_k (n + 1 \pm \sqrt{z/5})_k (n \pm \sqrt{z \pm 2\sqrt{c}})_k}{(n + 1/2)_k (n \pm \sqrt{z/5})_k (-4)^k \prod_{j=n+1}^{n+k} (j^4 - zj^2 + c)} \\
&= -\frac{(4n^2 - c/n^2)(n-1)!}{n^2 (3/2)_{n-1} (-4)^{n-1}} \prod_{j=1}^{n-1} \frac{(j^2 - n^2 - c/n^2)^2 - 4c}{j^4 - j^2 n^2 - cj^2/n^2 + c} \\
&\quad \times {}_7F_6 \left( \begin{matrix} n, n + 1 \pm \sqrt{(n^4 + c)/5n^2}, 2n \pm \sqrt{c}/n, \pm \sqrt{c}/n \\ n + 1/2, n \pm \sqrt{(n^4 + c)/5n^2}, 2n + 1, n + 1 \pm \sqrt{c}/n \end{matrix} \middle| -\frac{1}{4} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&{}_7F_6 \left( \begin{matrix} n, n + 1 \pm \sqrt{(n^4 + c)/5n^2}, 2n \pm \sqrt{c}/n, \pm \sqrt{c}/n \\ n + 1/2, n \pm \sqrt{(n^4 + c)/5n^2}, 2n + 1, n + 1 \pm \sqrt{c}/n \end{matrix} \middle| -\frac{1}{4} \right) \\
&= \frac{n4^n (3/2)_{n-1}}{(4n^2 - c/n^2)(n-1)!} \prod_{j=1}^{n-1} \frac{n^2 j^2 + cj^2/n^2 - j^4 - c}{n^4 + j^4 + c^2/n^4 - 2c - 2j^2 n^2 - 2j^2 c/n^2} \\
&= \frac{2n^2}{4n^2 - c/n^2} \left( \frac{2n}{n} \right) \prod_{j=1}^{n-1} \frac{(n^2 - j^2)(j^2 - c/n^2)}{((n+j)^2 - c/n^2)((n-j)^2 - c/n^2)} \\
&= \frac{1}{4n^2 - c/n^2} \left( \frac{(2n)!}{n!} \right)^2 \prod_{j=1}^{n-1} \frac{j^2 - c/n^2}{((n+j)^2 - c/n^2)((n-j)^2 - c/n^2)} \\
&= \frac{1}{4n^2 - c/n^2} \left( \frac{(2n)!}{n!} \right)^2 \prod_{j=1}^{n-1} \frac{1}{(n+j)^2 - c/n^2} \\
&= \left( \frac{(2n)!}{n!} \right)^2 \prod_{j=n+1}^{2n} (j^2 - c/n^2)^{-1} = \prod_{j=n+1}^{2n} j^2 (j^2 - c/n^2)^{-1} \\
&= \prod_{j=n+1}^{2n} \left( 1 - \frac{c}{j^2 n^2} \right)^{-1}.
\end{aligned}$$

□

## 5. ASYMPTOTICS AND VALUES AT THE NEGATIVE INTEGERS

Here we derive related formulae for values of the Riemann zeta function at the negative integers by comparing asymptotic expansions.

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