Depth reduction of a class of Witten zeta functions

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Abstract

We show that if a, b, c, d, f are positive integers such that $a+b+c+d+f$ is even, then the Witten zeta value $\zeta_{\mathfrak{sl}(4)}(a, b, c, d, 0, f)$ is expressible in terms of Witten zeta functions with fewer arguments.

1 Introduction

Let $\mathbf{N}$ be the set of positive integers, $\mathbf{Q}$ the field of rational numbers, $\mathbf{C}$ the field of complex numbers.

For any semisimple Lie algebra $\mathfrak{g}$, the Witten zeta function(cf. [5]) is defined by

$$\zeta_{\mathfrak{g}}(s) = \sum_{\rho} (\dim\, \rho)^{-s},$$

where $s \in \mathbf{C}$ and ρ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. In order the calculate the volumes of certain moduli space, Witten [7] introduced the values

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$\zeta_{\mathfrak{g}}(2k)$ for $k \in \mathbf{N}$ and showed that $\pi^{-2kl}\zeta_{\mathfrak{g}}(2k) \in \mathbf{Q}$, where l is the number of positive roots of $\mathfrak{g}$.

For positive integer r, Matsumoto and Tsumura [5] defined a multi-variate extension, called the Witten multiple zeta-function associated with $\mathfrak{sl}(r+1)$, by

$$\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s}) = \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} \prod_{k=1}^{r-j+1} \left(\sum_{v=k}^{j+k-1} m_v \right)^{-s_{j,k}} \tag{1}$$

where

$$\mathbf{s} = (s_{j,k})_{1\le j\le r;\ 1\le k\le r-j+1} \in \mathbf{C}^{r(r+1)/2}, \quad \Re(s_{j,k}) > 1.$$

In particular ([5], section 2, Prop 2.1), if $m \in \mathbf{N}$ we denote

$$\zeta_{\mathfrak{sl}(r+1)}(2m) := \prod_{1\le j<k\le r+1} (k-j)\zeta_{\mathfrak{sl}(r+1)}(\underbrace{2m,\ldots,2m}_{r(r+1)/2}).$$

As in [1], given the Witten multiple zeta-function (1), we define the *depth* to be r. Further, if the zeta functions $y_1,\ldots,y_k$ have depth $r_1,\ldots,r_k$ respectively, then for $a_1,\ldots,a_k \in \mathbf{C}$, we define the *depth* of $a_1y_1 + \cdots + a_ky_k$ to be $\max\{r_i : 1 \le i \le k\}$. We would like to know which sums can be expressed in terms of lower depth sums. When a sum can be so expressed, we say it is *reducible*.

An explicit evaluation for $\zeta_{\mathfrak{sl}(3)}(2m)$ ($m \in \mathbf{N}$) was independently discovered by D. Zagier, S. Garoufalidis, and L. Weinstein (see [8, page 506]). In [3], Gunnells and Sczech provided a generalization of the continued-fraction algorithm to compute high-dimensional Dedekind sums. As examples, they gave explicit evaluations of $\zeta_{\mathfrak{sl}(3)}(2m)$ and $\zeta_{\mathfrak{sl}(4)}(2m)$. Matsumoto and Tsumura [5] considered functional relations for Witten multiple zeta-functions, and found that

$$\begin{aligned}(-1)^a\zeta_{\mathfrak{sl}(4)}(s_1,s_2,a,s_3,0,b) + (-1)^b\zeta_{\mathfrak{sl}(4)}(s_1,s_2,b,s_3,0,a)&\\ + \zeta_{\mathfrak{sl}(4)}(a,0,s_2,s_1,b,s_3) + \zeta_{\mathfrak{sl}(4)}(b,0,s_1,s_2,a,s_3)&\end{aligned} \tag{2}$$

is reducible for any $a,b \in \mathbf{N}$ and $s_1,s_2,s_3 \in \mathbf{C}$.

In this paper, we provide a combinatorial method which gives a simpler formula for the quantity (2). Furthermore, we show that if a,b,c,d,f are positive integers such that $a+b+c+d+f$ is even, then $\zeta_{\mathfrak{sl}(4)}(a,b,c,d,0,f)$ is reducible.

2 Functional relation

Lemma 2.1. *If the function $F : \mathbf{Z}_{\ge 0} \times \mathbf{Z}_{\ge 0} \times \mathbf{C} \to \mathbf{C}$ has the property that there exist $p,\ q \in \mathbf{C}$ such that for every $a,b \in \mathbf{N}$ and every $s \in \mathbf{C}$ the relation*

$$F(a,b,s) = pF(a-1,b,s+1) + qF(a,b-1,s+1)$$

holds, then for every $a, b \in \mathbf{N}$ *and every* $s \in \mathbf{C}$*,*

$$F(a,b,s) = \sum_{j=1}^{b} p^a q^{b-j} \binom{a+b-j-1}{a-1} F(0,j,a+b+s-j) + \sum_{j=1}^{a} p^{a-j} q^b \binom{a+b-j-1}{b-1} F(j,0,a+b+s-j). \tag{3}$$

Proof. It's easy to prove Lemma 2.1 by induction. □

The Euler sum of depth r and weight w is a multiple series of the form

$$\zeta(s_1,\ldots,s_r) := \sum_{n_1>\cdots>n_r>0} \prod_{j=1}^{r} n_j^{-s_j}, \tag{4}$$

with *weight* $w := s_1+\cdots+s_r$. Now let's recall the following result concerning the reduction on the triple Euler sums.

Lemma 2.2 (Borwein-Girgensohn [2])**.** *Let* a, b, c *be positive integers. If* $a+b+c$ *is even or less than or equal to 10, then* $\zeta(a,b,c)$ *can be expressed as a rational linear combination of products of single and double Euler sums of weight* $a+b+c$*.*

Lemma 2.3 (Huard-Williams-Zhang [4])**.** *If* a, b, c *be positive integers, then*

$$\zeta_{\mathfrak{sl}(3)}(a,b,c) = \left\{ \sum_{j=1}^{a} \binom{a+b-j-1}{b-1} + \sum_{j=1}^{b} \binom{a+b-j-1}{a-1} \right\} \zeta(a+b+c-j, j). \tag{5}$$

Moreover, $\zeta_{\mathfrak{sl}(3)}(a,b,c)$ *can be explicitly evaluated in terms of the values of Riemann zeta functions when* $a+b+c$ *is odd.*

Theorem 2.1. *If* $a, b \in \mathbf{N}$*, then*

$$\begin{aligned}
&(-1)^a \zeta_{\mathfrak{sl}(4)}(s_1,s_2,a,s_3,0,b) + (-1)^b \zeta_{\mathfrak{sl}(4)}(s_1,s_2,b,s_3,0,a) \\
&\qquad + \zeta_{\mathfrak{sl}(4)}(a,0,s_2,s_1,b,s_3) + \zeta_{\mathfrak{sl}(4)}(b,0,s_1,s_2,a,s_3) \\
&= \sum_{i=1}^{\max(a,b)} \left\{ \binom{a+b-i-1}{a-1} + \binom{a+b-i-1}{b-1} \right\} (-1)^i \zeta(i) \\
&\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b-i) \\
&\quad + \sum_{i=1}^{a} \binom{a+b-i-1}{b-1} \Big\{ \zeta(i) \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b-i) \\
&\qquad - \zeta_{\mathfrak{sl}(3)}(s_1+i,s_2,s_3+a+b-i) - \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b) \Big\}
\end{aligned}$$

$$+\sum_{i=1}^{b}\binom{a+b-i-1}{a-1}\Big\{\zeta(i)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b-i)$$
$$-\zeta_{\mathfrak{sl}(3)}(s_2+i,s_1,s_3+a+b-i)-\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b)\Big\}. \tag{6}$$

Proof. From the definition (1) of the Witten multiple zeta-function, we have

$$\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)=\zeta_{\mathfrak{sl}(4)}(s_3,s_2,s_1,s_5,s_4,s_6). \tag{7}$$

Next, for any $a,b\in\mathbf{N}$ and $s_1,s_2,s_3\in\mathbf{C}$, since

$$\zeta_{\mathfrak{sl}(4)}(s_1,s_2,a,s_3,0,b)=\zeta_{\mathfrak{sl}(4)}(s_1,s_2,a,s_3+1,0,b-1)-\zeta_{\mathfrak{sl}(4)}(s_1,s_2,a-1,s_3+1,0,b),$$

by Lemma 2.1, we have

$$\zeta_{\mathfrak{sl}(4)}(s_1,s_2,a,s_3,0,b)=\sum_{i=1}^{a}\binom{a+b-i-1}{b-1}(-1)^{a+i}\zeta_{\mathfrak{sl}(4)}(s_1,s_2,i,s_3+a+b-i,0,0)$$
$$+\sum_{i=1}^{b}\binom{a+b-i-1}{a-1}(-1)^{a}\zeta_{\mathfrak{sl}(4)}(s_1,s_2,0,s_3+a+b-i,0,i). \tag{8}$$

Similarly, we have

$$\zeta_{\mathfrak{sl}(4)}(s_1,s_2,b,s_3,0,a)=\sum_{i=1}^{b}\binom{a+b-i-1}{a-1}(-1)^{b+i}\zeta_{\mathfrak{sl}(4)}(s_1,s_2,i,s_3+a+b-i,0,0)$$
$$+\sum_{i=1}^{a}\binom{a+b-i-1}{b-1}(-1)^{b}\zeta_{\mathfrak{sl}(4)}(s_1,s_2,0,s_3+a+b-i,0,i), \tag{9}$$

$$\zeta_{\mathfrak{sl}(4)}(a,0,s_2,s_1,b,s_3)=\sum_{i=1}^{a}\binom{a+b-i-1}{b-1}\zeta_{\mathfrak{sl}(4)}(i,0,s_2,s_1,0,s_3+a+b-i)$$
$$+\sum_{i=1}^{b}\binom{a+b-i-1}{a-1}\zeta_{\mathfrak{sl}(4)}(0,0,s_2,s_1,i,s_3+a+b-i), \tag{10}$$

and

$$\zeta_{\mathfrak{sl}(4)}(b,0,s_1,s_2,a,s_3)=\sum_{i=1}^{b}\binom{a+b-i-1}{a-1}\zeta_{\mathfrak{sl}(4)}(i,0,s_1,s_2,0,s_3+a+b-i)$$
$$+\sum_{i=1}^{a}\binom{a+b-i-1}{b-1}\zeta_{\mathfrak{sl}(4)}(0,0,s_1,s_2,i,s_3+a+b-i). \tag{11}$$

Since

$$\zeta_{\mathfrak{sl}(4)}(a,b,c,d,0,0) = \zeta(c)\zeta_{\mathfrak{sl}(3)}(a,b,d), \tag{12}$$

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(a,b,0,c,0,d) &= \sum_{\substack{n_1,n_2=1\\ v>n_1+n_2}} \frac{1}{v^d n_1^a n_2^b (n_1+n_2)^c}\\ &= \sum_{\substack{n_1,n_2=1\\ v>n_1+n_2}} \frac{1}{v^d n_1^b n_2^a (n_1+n_2)^c},\end{aligned} \tag{13}$$

$$\zeta_{\mathfrak{sl}(4)}(a,0,b,c,0,d) = \sum_{\substack{n_1,n_2=1\\ v<n_1}} \frac{1}{v^a n_1^c n_2^b (n_1+n_2)^d}, \tag{14}$$

$$\zeta_{\mathfrak{sl}(4)}(0,0,a,b,c,d) = \sum_{\substack{n_1,n_2=1\\ n_1+n_2>v>n_1}} \frac{1}{v^c n_1^a n_2^b (n_1+n_2)^d}, \tag{15}$$

we find that

$$\begin{aligned}&\zeta_{\mathfrak{sl}(4)}(s_1,s_2,0,s_3+a+b-i,0,i) + \zeta_{\mathfrak{sl}(4)}(i,0,s_2,s_1,0,s_3+a+b-i)\\ &\qquad + \zeta_{\mathfrak{sl}(4)}(0,0,s_1,s_2,i,s_3+a+b-i)\\ &= \zeta(i)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b-i) - \zeta_{\mathfrak{sl}(3)}(s_1+i,s_2,s_3+a+b-i)\\ &\qquad - \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b)\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\zeta_{\mathfrak{sl}(4)}(s_1,s_2,0,s_3+a+b-i,0,i) + \zeta_{\mathfrak{sl}(4)}(i,0,s_1,s_2,0,s_3+a+b-i)\\ &\qquad + \zeta_{\mathfrak{sl}(4)}(0,0,s_2,s_1,i,s_3+a+b-i)\\ &= \zeta(i)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b-i) - \zeta_{\mathfrak{sl}(3)}(s_2+i,s_1,s_3+a+b-i)\\ &\qquad - \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+a+b)\end{aligned} \tag{17}$$

Now combining equations (8-17), we complete the proof. □

Lemma 2.4. *Every Witten multiple zeta value of the form $\zeta_{\mathfrak{sl}(4)}(a,b,1,d,0,1)$ with $a,b,d \in$ $\mathbf{N}$ can be expressed as a rational linear combination of products of single and double Euler sums when $a+b+d$ is even or $a+b+d \le 8$.*

Proof.

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(a,b,1,d,0,1) &= \sum_{i=1}^{a} \binom{a+b-i-1}{b-1} \zeta_{\mathfrak{sl}(4)}(i,0,1,a+b+d-i,0,1)\\ &\quad + \sum_{i=1}^{b} \binom{a+b-i-1}{a-1} \zeta_{\mathfrak{sl}(4)}(0,i,1,a+b+d-i,0,1).\end{aligned} \tag{18}$$

However, for any $a, d \in \mathbf{N}$,

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(a,0,1,d,0,1) &= \zeta_{\mathfrak{sl}(4)}(0,a,1,d,0,1) \\ &= \zeta_{\mathfrak{sl}(4)}(a,0,1,0,0,d+1) + \sum_{i=1}^{d} \zeta(d+2-i,i,a),\end{aligned} \tag{19}$$

and

$$\zeta_{\mathfrak{sl}(4)}(a,0,1,0,0,d+1) = \zeta(d+1,a,1) + \sum_{i=1}^{a} \zeta(d+1,a+1-i,i). \tag{20}$$

We complete the proof by combining this with Lemma 2.2. ☐

Theorem 2.2. *Every Witten multiple zeta value of the form $\zeta_{\mathfrak{sl}(4)}(a,b,c,d,0,f)$ with $a,b,c,d,f, \in \mathbf{N}$ can be expressed as a rational linear combination of products of single and double Euler sums when $a+b+c+d+f$ is even or $a+b+c+d+f \leq 10$.*

Proof. From Lemma 2.1, we see that

$$\begin{aligned}&\frac{1}{n_1^a n_2^b n_3^c (n_1+n_2)^d (n_1+n_2+n_3)^f} \\ &\qquad = \sum_{i=1}^{c} \binom{c+f-i-1}{f-1} (-1)^{c+i} \frac{1}{n_1^a n_2^b n_3^i (n_1+n_2)^{c+d+f-i}} \\ &\qquad + \sum_{i=1}^{f} \binom{c+f-i-1}{c-1} (-1)^{c} \frac{1}{n_1^a n_2^b (n_1+n_2)^{c+d+f-i} (n_1+n_2+n_3)^i}.\end{aligned} \tag{21}$$

Also

$$\begin{aligned}&\frac{1}{n_1^a n_2^b (n_1+n_2)^{c+d+f-i} (n_1+n_2+n_3)^i} \\ &\qquad = \sum_{j=1}^{a} \binom{a+b-j-1}{b-1} \frac{1}{n_1^j (n_1+n_2)^{a+b+c+d+f-i-j} (n_1+n_2+n_3)^i} \\ &\qquad + \sum_{j=1}^{b} \binom{a+b-j-1}{a-1} \frac{1}{n_2^j (n_1+n_2)^{a+b+c+d+f-i-j} (n_1+n_2+n_3)^i}.\end{aligned} \tag{22}$$

Now combine (20), (21) and Lemma 2.4 and sum over all ordered triples of positive integers (n_1, n_2, n_3) to obtain

$$\zeta_{\mathfrak{sl}(4)}(a,b,c,d,0,f) = \sum_{i=2}^{c} \binom{c+f-i-1}{f-1} (-1)^{c+i} \zeta(i) \zeta_{\mathfrak{sl}(3)}(a,b,c+d+f-i)$$

$$
\begin{aligned}
&+\sum_{i=2}^{f}\binom{c+f-i-1}{c-1}(-1)^c\Bigg\{\sum_{j=1}^{a}\binom{a+b+j-1}{b-1}\\
&\qquad\times\zeta(i,c+d+f+a+b-i-j,j)\\
&+\sum_{j=1}^{b}\binom{a+b+j-1}{a-1}\zeta(i,c+d+f+a+b-i-j,j)\Bigg\}\\
&-(-1)^c\binom{c+f-2}{c-1}\zeta_{\mathfrak{sl}(4)}(a,b,1,c+d+f-2,0,1). \qquad (23)
\end{aligned}
$$

By Lemmas 2.2, 2.3 and 2.4, we complete the proof. □

Remark. When $d=0$, the Witten zeta value $\zeta_{\mathfrak{sl}(4)}(a,b,c,0,0,f)$ can also be viewed as a Mordell-Tornheim sum with depth 3. The fact that every such sum can be expressed as a rational linear combination of products of single and double Euler sums when the weight $a+b+c+f$ is even has been shown in [6] and [9].

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References

[1] J. M. Browein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Special values of multiple polylogarithms, *Trans. Amer. Math. Soc.* **353** (2001), no. 3 p. 907-941.

[2] J. M. Borwein and R. Girgensohn, Evaluations of triple Euler sums, *Electron. J. Combin.*, **3** (1996), no. 1, Research Paper 23, approx. 27 pp.

[3] P. E. Gunnells and R. Sczech, Evaluations of Dedekind sums, Eisenstein cocycles, and special values of L-functions, *Duke. J. Math.*, **118** (2003), p. 229-260.

[4] J. G. Huard, K. S. Williams and N. Y. Zhang, On Tornheim's double series, Acta Arith., **75** (1996), no 2, 105–117. [MR 1379394] (97f:11073)

[5] K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie Algebras I, *Ann. Inst. Fourier*, **56**(5) (2006), p. 1457-1504.

[6] H. Tsumura, On Mordell-Tornheim zeta values, *Proc. Amer. Math. Soc.*, **133** (2005), no. 8, 2387–2393. [MR 2138881] (2006k:11179)

[7] E. Witten, On quantum gauge theories in two dimensions, *Comm. Math. Phy.* **141 (1991)**: 153-209.

[8] D. Zagier, Values of zeta functions and their applications, in *Proc. First Congress of Math.,Paris, vol. II* Progress in Math., vol. 120, Birkhäuser, 1994, p. 497-512.

[9] X. Zhou, D. M. Bradley and T. Cai, On Mordell-Tornheim sums and multiple zeta values, submitted.