

# A SIGNED ANALOG OF EULER'S REDUCTION FORMULA FOR THE DOUBLE ZETA FUNCTION

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ABSTRACT. The double zeta function is a function of two arguments defined by a double Dirichlet series, and was first studied by Euler in response to a letter from Goldbach in 1742. By calculating many examples, Euler inferred a closed form evaluation of the double zeta function in terms of values of the Riemann zeta function, in the case when the two arguments are positive integers with opposite parity. Here, we consider a signed analog of Euler's evaluation: namely a reduction formula for the signed double zeta function that reduces to Euler's evaluation when the signs are specialized to 1. This formula was first stated in a 1997 paper by Borwein, Bradley and Broadhurst and was subsequently proved by Flajolet and Salvy using contour integration. The purpose here is to give an elementary proof based on a partial fraction identity.

## 1. INTRODUCTION

The double zeta function is defined by

$$\zeta(s, t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \quad \Re(s) > 1, \quad \Re(s+t) > 2. \quad (1)$$

The problem of evaluating sums of the form (1) with integers  $s > 1$  and  $t > 0$  seems to have been first proposed in a letter from Goldbach to Euler [5] in 1742. (See also [4, 7] and [1, p. 253].) Calculating several examples led Euler to infer a closed form evaluation of the double zeta function in terms of values of the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,$$

in the case when  $s-1$  and  $t-1$  are positive integers with opposite parity. Euler's evaluation can be expressed as follows. Let  $s > 1$  and  $t > 1$  be integers with opposite parity (i.e.  $s+t$  is odd) and let  $2m = \max(s, t)$ . Then

$$\begin{aligned} \zeta(s, t) = & \frac{1}{2} \left( (1 + (-1)^s) \zeta(s) \zeta(t) + \frac{1}{2} \left[ (-1)^s \binom{s+t}{s} - 1 \right] \zeta(s+t) \right) \\ & + (-1)^{s+1} \sum_{k=1}^m \left[ \binom{s+t-2k-1}{t-1} + \binom{s+t-2k-1}{s-1} \right] \zeta(2k) \zeta(s+t-2k). \quad (2) \end{aligned}$$

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The formula (2) is also valid when  $t = 1$  and  $s$  is even, but that case is subsumed by another formula of Euler, namely

$$\zeta(s, 1) = \frac{1}{2}s\zeta(s+1) - \frac{1}{2}\sum_{k=2}^{s-1}\zeta(k)\zeta(s+1-k), \quad (3)$$

which is valid for all integers  $s > 1$ . In [2], Borwein, Bradley and Broadhurst considered the more general Euler sum

$$\zeta(s_1, s_2, \dots, s_k; \sigma_1, \sigma_2, \dots, \sigma_k) := \sum_{n_1 > n_2 > \dots > n_k > 0} \prod_{j=1}^k \sigma_j^{n_j} n_j^{-s_j} \quad (4)$$

with each  $\sigma_j \in \{-1, 1\}$ . Among the many other results for (4) listed therein is an explicit formula for the case  $k = 2$  that reduces to (2) when  $\sigma_1 = \sigma_2 = 1$ . We restate this result as follows:

**Proposition 1.** *Let  $\sigma, \tau \in \{-1, 1\}$ , and let  $s$  and  $t$  be positive integers such that  $s+t$  is odd,  $s > (1+\sigma)/2$ , and  $t > (1+\tau)/2$ . Then*

$$\begin{aligned} \zeta(s, t; \sigma, \tau) &= \frac{1}{2}(1 + (-1)^s)\zeta(s; \sigma)\zeta(t; \tau) - \frac{1}{2}\zeta(s+t; \sigma\tau) \\ &\quad + (-1)^t \sum_{0 \leq k \leq t/2} \binom{s+t-2k-1}{s-1} \zeta(2k; \sigma\tau)\zeta(s+t-2k; \sigma) \\ &\quad + (-1)^t \sum_{0 \leq k \leq s/2} \binom{s+t-2k-1}{t-1} \zeta(2k; \sigma\tau)\zeta(s+t-2k; \tau). \end{aligned} \quad (5)$$

In Proposition 1, it is understood that  $\zeta(0; \sigma\tau) = -1/2$  in accordance with the analytic continuation of  $s \mapsto \zeta(s; \sigma\tau)$ . The restriction  $t > (1+\tau)/2$  can be removed if in (5) we interpret  $\zeta(1; 1) = 0$  wherever it occurs. That is, if  $\sigma \in \{-1, 1\}$  and  $s$  is an even positive integer, then

$$\zeta(s, 1; \sigma, 1) = \frac{1}{2}(s-1)\zeta(s+1; \sigma) + \frac{1}{2}\zeta(s+1) - \sum_{k=1}^{(s/2)-1} \zeta(2k; \sigma)\zeta(s+1-2k). \quad (6)$$

Note that by (3) we know that the case  $\sigma = 1$  of (6) can be extended to *all* integers  $s > 1$ , not just even  $s$ .

Using contour integration, Flajolet and Salvy [6] proved an equivalent version of Proposition 1. Our intention here is to give an elementary proof based on the partial fraction decomposition

$$\frac{1}{x^s y^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a}(x+y)^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{(x+y)^{s+a} y^{t-a}}, \quad (7)$$

which is valid for positive integers  $s$  and  $t$  and non-zero real numbers  $x$  and  $y$  such that  $x+y \neq 0$ . As in [3], we note that (7) is readily proved by applying the partial differential operator

$$\frac{1}{(r-1)!} \left( -\frac{\partial}{\partial x} \right)^{r-1} \frac{1}{(s-1)!} \left( -\frac{\partial}{\partial y} \right)^{s-1}$$

to both sides of the identity

$$\frac{1}{xy} = \frac{1}{x+y} \left( \frac{1}{x} + \frac{1}{y} \right).$$

## 2. PROOF OF PROPOSITION 1

**Definition 1.** Let  $N$  be a positive integer and let  $s, t, \sigma, \tau$  be complex numbers. Define

$$\zeta_N(s, t; \sigma, \tau) := \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\sigma^n \tau^k}{n^s k^t} = (-1)^t \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\sigma^n \tau^{n-k}}{n^s (k-n)^t} \quad \text{and} \quad \zeta_N(s; \sigma) = \sum_{n=1}^N \frac{\sigma^n}{n^s}.$$

Now suppose that  $s$  and  $t$  are positive integers. In (7) let  $x = n$ ,  $y = k - n$ , multiply through by  $(-1)^t \sigma^n \tau^{n-k}$  and sum over all positive integers  $n$  and  $k$  satisfying  $N > n > k > 0$ . We find that

$$\begin{aligned} & (-1)^t \zeta_N(s, t; \sigma, \tau) \\ &= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\sigma^n \tau^{n-k}}{n^{s-a} k^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\sigma^n \tau^{n-k}}{k^{s+a} (k-n)^{t-a}} \\ &= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \left\{ \sum_{n=1}^N \sum_{k=1}^N \frac{\sigma^n \tau^{n-k}}{n^{s-a} k^{t+a}} - \sum_{k=1}^N \frac{\sigma^k}{k^{s+t}} - \sum_{k=1}^N \sum_{n=1}^{k-1} \frac{\sigma^n \tau^{n-k}}{n^{s-a} k^{t+a}} \right\} \\ &+ (-1)^t \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} (-1)^a \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\sigma^n \tau^{n-k}}{k^{s+a} (n-k)^{t-a}} \\ &= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} [\zeta_N(s-a; \sigma\tau) \zeta_N(t+a; 1/\tau) - \zeta_N(t+a, s-a; 1/\tau, \sigma\tau)] \\ &- \binom{s+t-1}{s-1} \zeta_N(s+t; \sigma) + (-1)^t \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+a} m^{t-a}}. \end{aligned}$$

It follows that

$$\begin{aligned} & (-1)^t \zeta_N(s, t; \sigma, \tau) + (-1)^s \zeta_N(t, s; \tau, \sigma) \\ &= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta_N(s-a; \sigma\tau) \zeta_N(t+a; 1/\tau) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta_N(t-a; \sigma\tau) \zeta_N(s+a; 1/\sigma) \\ &- \left[ \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta_N(t+a, s-a; 1/\tau, \sigma\tau) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta_N(s+a, t-a; 1/\sigma, \sigma\tau) \right] \\ &- \binom{s+t-1}{s-1} \zeta_N(s+t; \sigma) - \binom{s+t-1}{t-1} \zeta_N(s+t; \tau) \\ &+ (-1)^t \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+a} m^{t-a}} \\ &+ (-1)^s \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\tau^{m+k} \sigma^m}{k^{t+a} m^{s-a}}. \end{aligned} \tag{8}$$

By (7) again,

$$\zeta_N(s; 1/\sigma) \zeta_N(t; 1/\tau) = \sum_{x=1}^N \sum_{y=1}^N \frac{\sigma^{-x} \tau^{-y}}{x^s y^t}$$

$$\begin{aligned}
&= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \sum_{x=1}^N \sum_{y=1}^N \frac{\sigma^{-x} \tau^{-y}}{x^{s-a} (x+y)^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \sum_{x=1}^N \sum_{y=1}^N \frac{\sigma^{-x} \tau^{-y}}{(x+y)^{s+a} y^{t-a}} \\
&= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \left[ \sum_{n=1}^N \sum_{x=1}^{n-1} \frac{\sigma^{-x} \tau^{x-n}}{x^{s-a} n^{t+a}} + \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^{-x} \tau^{x-n}}{x^{s-a} n^{t+a}} \right] \\
&+ \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \left[ \sum_{n=1}^N \sum_{y=1}^{n-1} \frac{\sigma^{y-n} \tau^{-y}}{n^{s+a} y^{t-a}} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y-n} \tau^{-y}}{n^{s+a} y^{t-a}} \right] \\
&= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \left[ \zeta_N(t+a, s-a; 1/\tau, \tau/\sigma) + \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^{-x} \tau^{x-n}}{x^{s-a} n^{t+a}} \right] \\
&+ \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \left[ \zeta_N(s+a, t-a; 1/\sigma, \sigma/\tau) + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y-n} \tau^{-y}}{n^{s+a} y^{t-a}} \right].
\end{aligned}$$

Rearranging this yields

$$\begin{aligned}
&\sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta_N(t+a, s-a; 1/\tau, \tau/\sigma) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta_N(s+a, t-a; 1/\sigma, \sigma/\tau) \\
&= \zeta_N(s; 1/\sigma) \zeta_N(t; 1/\tau) - \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^{-x} \tau^{x-n}}{x^{s-a} n^{t+a}} \\
&\quad - \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y-n} \tau^{-y}}{n^{s+a} y^{t-a}}. \quad (9)
\end{aligned}$$

Henceforth assume that  $\sigma \in \{-1, 1\}$  and  $\tau \in \{-1, 1\}$ . Then  $\sigma = 1/\sigma$ ,  $\tau = 1/\tau$  and from (8) and (9) we infer that

$$\begin{aligned}
&(-1)^t \zeta_N(s, t; \sigma, \tau) + (-1)^s \zeta_N(t, s; \tau, \sigma) \\
&= \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta_N(s-a; \sigma\tau) \zeta_N(t+a; \tau) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta_N(t-a; \sigma\tau) \zeta_N(s+a; \sigma) \\
&- \zeta_N(s; \sigma) \zeta_N(t; \tau) - \binom{s+t-1}{s-1} \zeta_N(s+t; \sigma) - \binom{s+t-1}{t-1} \zeta_N(s+t; \tau) \\
&+ (-1)^t \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+a} m^{t-a}} + (-1)^s \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\tau^{m+k} \sigma^m}{k^{t+a} m^{s-a}} \\
&+ \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^x \tau^{x+n}}{x^{s-a} n^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n} \tau^y}{n^{s+a} y^{t-a}}. \quad (10)
\end{aligned}$$

But if  $N$  is any positive integer, then

$$\begin{aligned}
\zeta_N(s; \sigma) \zeta_N(t; \tau) &= \sum_{x=1}^N \sum_{y=1}^N \frac{\sigma^x \tau^y}{x^s y^t} = \sum_{x=1}^N \sum_{y=1}^{x-1} \frac{\sigma^x \tau^y}{x^s y^t} + \sum_{x=y=1}^N \frac{\sigma^x \tau^y}{x^s y^t} + \sum_{y=1}^N \sum_{x=1}^{y-1} \frac{\sigma^x \tau^y}{x^s y^t} \\
&= \zeta_N(s, t; \sigma, \tau) + \zeta_N(s+t; \sigma\tau) + \zeta_N(t, s; \tau, \sigma),
\end{aligned}$$

whence

$$\zeta_N(t, s; \tau, \sigma) = \zeta_N(s; \sigma) \zeta_N(t; \tau) - \zeta_N(s+t; \sigma\tau) - \zeta_N(s, t; \sigma, \tau).$$

If we use this in (10) and multiply through by  $(-1)^t$  there comes

$$\begin{aligned}
& \zeta_N(s, t; \sigma, \tau) + (-1)^{s+t} [\zeta_N(s; \sigma) \zeta_N(t; \tau) - \zeta_N(s+t; \sigma\tau) - \zeta_N(s, t; \sigma, \tau)] \\
&= (-1)^t \left[ \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta_N(s-a; \sigma\tau) \zeta_N(t+a; \tau) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta_N(t-a; \sigma\tau) \zeta_N(s+a; \sigma) \right] \\
&+ (-1)^{t+1} \left[ \zeta_N(s; \sigma) \zeta_N(t; \tau) + \binom{s+t-1}{s-1} \zeta_N(s+t; \sigma) + \binom{s+t-1}{t-1} \zeta_N(s+t; \tau) \right] \\
&+ \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+a} m^{t-a}} + (-1)^{s+t} \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} (-1)^a \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\tau^{m+k} \sigma^m}{k^{t+a} m^{s-a}} \\
&+ (-1)^t \left[ \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^x \tau^{x+n}}{x^{s-a} n^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n} \tau^y}{n^{s+a} y^{t-a}} \right].
\end{aligned}$$

Now assume also that  $s+t$  is odd. Writing  $(-1)^s$  for  $(-1)^{t+1}$  and re-indexing the sums yields

$$\begin{aligned}
2\zeta_N(s, t; \sigma, \tau) &= (1 + (-1)^s) \zeta_N(s; \sigma) \zeta_N(t; \tau) - \zeta_N(s+t; \sigma\tau) \\
&+ (-1)^t \sum_{b=1}^s \binom{s+t-b-1}{t-1} \left[ \zeta_N(b; \sigma\tau) \zeta_N(s+t-b; \tau) \right. \\
&\quad \left. + (-1)^b \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\tau^{m+k} \sigma^m}{k^{s+t-b} m^b} + \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^x \tau^{x+n}}{x^b n^{s+t-b}} \right] \\
&+ (-1)^t \sum_{b=1}^t \binom{s+t-b-1}{s-1} \left[ \zeta_N(b; \sigma\tau) \zeta_N(s+t-b; \sigma) \right. \\
&\quad \left. + (-1)^b \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+t-b} m^b} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n} \tau^y}{n^{s+t-b} y^b} \right] \\
&- (-1)^t \binom{s+t-1}{s-1} \zeta_N(s+t; \sigma) - (-1)^t \binom{s+t-1}{t-1} \zeta_N(s+t; \tau). \tag{11}
\end{aligned}$$

Now suppose that  $b$  is odd and  $1 \leq b \leq t$ . Then

$$\begin{aligned}
& \left| \zeta_N(b; \sigma\tau) \zeta_N(s+t-b; \sigma) + (-1)^b \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+t-b} m^b} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n} \tau^y}{n^{s+t-b} y^b} \right| \\
&\leq \left| \sum_{k=1}^N \frac{\sigma^k}{k^{s+t-b}} \left[ \sum_{m=1}^N \frac{(\sigma\tau)^m}{m^b} - \sum_{m=1}^{N-k} \frac{(\sigma\tau)^m}{m^b} \right] \right| + \left| \sum_{y=1}^N \frac{(\sigma\tau)^y}{y^b} \sum_{n=N+1}^{N+y} \frac{\sigma^n}{n^{s+t-b}} \right| \\
&\leq \sum_{k=1}^N \frac{1}{k^{s+t-b}} \sum_{m=N-k+1}^N \frac{1}{m} + \sum_{y=1}^N \frac{1}{y} \sum_{n=N+1}^{N+y} \frac{1}{n^{s+t-b}} \\
&\leq \sum_{k=1}^N \frac{1}{k^{s+t-b}} \cdot \frac{k}{N-k+1} + \sum_{y=1}^N \frac{1}{y} \sum_{n=N+1}^{\infty} \frac{1}{n^{s+t-b-1} (n-1)} \\
&= \sum_{k=1}^N \frac{1}{k^{s+t-b-1} (N-k+1)} + \sum_{k=1}^N \frac{1}{k} \sum_{n=N}^{\infty} \frac{1}{n(n+1)^{s+t-b-1}}.
\end{aligned}$$

If  $t$  is also odd, then (recalling that  $s$  and  $t$  are positive integers with opposite parity)  $s$  is even and  $s + t - b - 1 \geq s - 1 \geq 1$ . On the other hand, if  $t$  is even, then  $b \leq t - 1$  since  $b$  is odd, and therefore  $s + t - b - 1 \geq s \geq 1$ . In either case,

$$\begin{aligned} & \left| \zeta_N(b; \sigma\tau) \zeta_N(s + t - b; \sigma) + (-1)^b \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+t-b} m^b} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n} \tau^y}{n^{s+t-b} y^b} \right| \\ & \leq \sum_{k=1}^N \frac{1}{k(N-k+1)} + \sum_{k=1}^N \frac{1}{k} \sum_{n=N}^{\infty} \frac{1}{n(n+1)} = \frac{1}{N+1} \sum_{k=1}^N \left( \frac{1}{k} + \frac{1}{N-k+1} \right) + \frac{1}{N} \sum_{k=1}^N \frac{1}{k} \leq \frac{3}{N} \sum_{k=1}^N \frac{1}{k} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ .

Next, suppose that  $1 < b \leq t$  and additionally  $s > (1 + \sigma)/2$ . Then

$$\lim_{N \rightarrow \infty} \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k} \tau^m}{k^{s+t-b} m^b} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma^{m+k} \tau^m}{k^{s+t-b} m^b} = \zeta(b; \sigma\tau) \zeta(s + t - b; \sigma)$$

and

$$\left| \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n} \tau^y}{n^{s+t-b} y^b} \right| \leq \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{1}{ny^2} = \sum_{y=1}^N \frac{1}{y^2} \sum_{n=N+1}^{N+y} \frac{1}{n} \leq \sum_{y=1}^N \frac{1}{y^2} \cdot \frac{y}{N+1} \rightarrow 0$$

as  $N \rightarrow \infty$ . Thus, under the hypotheses of Proposition 1, letting  $N \rightarrow \infty$  in (11) yields

$$\begin{aligned} \zeta(s, t; \sigma, \tau) &= \frac{1}{2} (1 + (-1)^s) \zeta(s; \sigma) \zeta(t; \tau) - \frac{1}{2} \zeta(s + t; \sigma\tau) \\ &+ (-1)^t \sum_{1 \leq k \leq s/2} \binom{s+t-2k-1}{t-1} \zeta(2k; \sigma\tau) \zeta(s+t-2k; \tau) \\ &+ (-1)^t \sum_{1 \leq k \leq t/2} \binom{s+t-2k-1}{s-1} \zeta(2k; \sigma\tau) \zeta(s+t-2k; \sigma) \\ &- \frac{1}{2} (-1)^t \left[ \binom{s+t-1}{s-1} \zeta(s+t; \sigma) + \binom{s+t-1}{t-1} \zeta(s+t; \tau) \right]. \end{aligned}$$

Since  $\zeta(0; \sigma\tau) = -1/2$  by definition, the final line can be absorbed into the two sums above by permitting  $k = 0$ .

### 3. THE CASE $t = \tau = 1$ : PROOF OF EQUATION (6)

Putting  $t = \tau = 1$  in (11) and noting that then  $s$  must be even yields

$$\begin{aligned} 2\zeta_N(s, 1; \sigma, 1) &= (s-1)\zeta_N(s+1; \sigma) + \zeta_N(s+1; 1) + \left( \zeta_N(s; \sigma) \zeta_N(1; 1) - \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^m}{km^s} \right) \\ &+ \left( \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k}}{k^s m} - \zeta_N(s; \sigma) \zeta_N(1; \sigma) \right) - \left( \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^x}{x^s n} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n}}{n^s y} \right) \\ &- \sum_{b=1}^{s-1} \left[ \zeta_N(b; \sigma) \zeta_N(s+1-b; 1) + (-1)^b \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^m}{k^{s+1-b} m^b} + \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^x}{x^b n^{s+1-b}} \right]. \end{aligned}$$

As in the proof of Proposition 1, we find that as  $N$  grows without bound, the expression in square brackets approaches zero when  $b$  is odd, and approaches  $2\zeta(b; \sigma)\zeta(s+1-b; 1)$  when  $b$  is even. To complete the

proof of equation (6), it suffices to show that the expressions in parentheses each tend to zero in the limit as  $N$  tends to infinity.

First, since  $s \geq 2$ ,

$$\begin{aligned} \left| \zeta_N(s; \sigma) \zeta_N(1; 1) - \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^m}{km^s} \right| &= \left| \sum_{k=1}^N \frac{1}{k} \sum_{m=N-k+1}^N \frac{\sigma^m}{m^s} \right| \leq \sum_{k=1}^N \frac{1}{k} \sum_{m=N-k+1}^N \frac{1}{m^2} \\ &= \sum_{m=1}^N \frac{1}{m^2} \sum_{k=N-m+1}^N \frac{1}{k} \leq \sum_{m=1}^N \frac{1}{m^2} \cdot \frac{m}{N-m+1} = \frac{1}{N+1} \sum_{m=1}^N \left( \frac{1}{m} + \frac{1}{N-m+1} \right) \\ &= \frac{2}{N+1} \sum_{m=1}^N \frac{1}{m} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ . Also,

$$\begin{aligned} \left| \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \frac{\sigma^{m+k}}{k^s m} - \zeta_N(s; \sigma) \zeta_N(1; \sigma) \right| &= \left| \sum_{k=1}^N \frac{\sigma^k}{k^s} \sum_{m=N-k+1}^N \frac{\sigma^m}{m} \right| \leq \sum_{k=1}^N \frac{1}{k^2} \sum_{m=N-k+1}^N \frac{1}{m} \\ &\leq \sum_{k=1}^N \frac{1}{k^2} \cdot \frac{k}{N-k+1} = \frac{1}{N+1} \sum_{k=1}^N \left( \frac{1}{k} + \frac{1}{N-k+1} \right) = \frac{2}{N+1} \sum_{k=1}^N \frac{1}{k} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ . Finally,

$$\begin{aligned} \left| \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{\sigma^x}{x^s n} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{\sigma^{y+n}}{n^s y} \right| &\leq \sum_{n=N+1}^{2N} \sum_{x=n-N}^N \frac{1}{x^2 n} + \sum_{n=N+1}^{2N} \sum_{y=n-N}^N \frac{1}{yn^2} \\ &= \sum_{x=1}^N \frac{1}{x^2} \sum_{n=N+1}^{N+x} \frac{1}{n} + \sum_{y=1}^N \frac{1}{y} \sum_{n=N+1}^{N+y} \frac{1}{n^2} \leq \sum_{x=1}^N \frac{1}{x^2} \cdot \frac{x}{N+1} + \sum_{y=1}^N \frac{1}{y} \sum_{n=N+1}^{\infty} \frac{1}{n(n-1)} \\ &= \frac{1}{N+1} \sum_{x=1}^N \frac{1}{x} + \frac{1}{N} \sum_{y=1}^N \frac{1}{y} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ .

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