

AN INFINITE SERIES FOR THE NATURAL LOGARITHM THAT CONVERGES THROUGHOUT ITS DOMAIN AND MAKES CONCAVITY TRANSPARENT

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ABSTRACT. The natural logarithm can be represented by an infinite series that converges for all positive real values of the variable, and which makes concavity patently obvious. Concavity of the natural logarithm is known to imply, among other things, the fundamental inequality between the arithmetic and geometric mean.

The inequality $\log x < x - 1$, which holds for all positive real $x \neq 1$ [2, Theorem 150, p. 106] is a fundamental but often overlooked property of the natural logarithm. In fact, it is equivalent to concavity. To see this, fix $a > 0$, replace x by x/a , and note that if $0 < x \neq a$, then

$$\log x - \log a = \log(x/a) < (x/a) - 1 = (x - a) \log' a,$$

which says that the graph of the natural logarithm lies below its tangent line at every point $(a, \log a)$. On the other hand, concavity of the natural logarithm is known to imply the ubiquitous inequality between the arithmetic and geometric mean of a finite set of positive real numbers [2, p. 119].

The inequality $\log x < x - 1$ for $0 < x \neq 1$ is an immediate consequence of the integral representation for the natural logarithm, via the identity

$$x - 1 - \log x = \int_1^x \int_1^t u^{-2} du dt,$$

since the double integral is obviously positive for all positive real $x \neq 1$. Nevertheless, it may be of interest to derive an alternative representation which, rather than employ integrals, expresses the difference $x - 1 - \log x$ as an infinite series each of whose terms is obviously positive for $0 < x \neq 1$.

To this end, we begin for $x > 0$ with the definition of the derivative

$$\log x = \left. \frac{d}{dh} x^h \right|_{h=0} = \lim_{h \rightarrow 0} \frac{x^h - 1}{h} = \lim_{h \rightarrow 0} h^{-1} (x^h - 1) = \lim_{n \rightarrow \infty} 2^n (x^{2^{-n}} - 1),$$

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as a limit of a difference quotient [1, Ch. III, §6, pp. 175–176]. Using this, we can express $x - 1 - \log x$ for $x > 0$ as a telescoping series:

$$\begin{aligned} x - 1 - \log x &= \lim_{n \rightarrow \infty} \sum_{k=1}^n [2^{k-1}(x^{2^{1-k}} - 1) - 2^k(x^{2^{-k}} - 1)] \\ &= \sum_{k=1}^{\infty} [2^{k-1}(x^{2^{-k}} - 1)(x^{2^{-k}} + 1) - 2^k(x^{2^{-k}} - 1)] \\ &= \sum_{k=1}^{\infty} 2^{k-1}(x^{2^{-k}} - 1)[(x^{2^{-k}} + 1) - 2] \\ &= \sum_{k=1}^{\infty} 2^{k-1}(x^{2^{-k}} - 1)^2. \end{aligned}$$

The series converges by limit comparison with the convergent geometric series $\sum_{k=1}^{\infty} 2^{-k}$, for if $x > 0$, then

$$\lim_{k \rightarrow \infty} 2^{k-1}(x^{2^{-k}} - 1)^2 / 2^{-k} = \lim_{k \rightarrow \infty} 2^{-k} 2^{k-1} \left(\frac{x^{2^{-k}} - 1}{2^{-k}} \right)^2 = \frac{1}{2} (\log x)^2.$$

Since each term of the series is clearly positive for $0 < x \neq 1$, the inequality $\log x < x - 1$ follows immediately.

REFERENCES

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