A q-ANALOG OF EULER'S DECOMPOSITION FORMULA FOR THE DOUBLE ZETA FUNCTION

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The double zeta function was first studied by Euler in response to a letter from Goldbach in 1742. One of Euler's results for this function is a decomposition formula, which expresses the product of two values of the Riemann zeta function as a finite sum of double zeta values involving binomial coefficients. Here, we establish a q-analog of Euler's decomposition formula. More specifically, we show that Euler's decomposition formula can be extended to what might be referred to as a "double q-zeta function" in such a way that Euler's formula is recovered in the limit as q tends to 1.

1. Introduction

The Riemann zeta function is defined for $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

Accordingly,

$$\zeta(s,t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \qquad \Re(s) > 1, \qquad \Re(s+t) > 2, \tag{1.2}$$

is known as the double zeta function. The sums (1.2), and more generally those of the form

$$\zeta(s_1, s_2, \ldots, s_m) := \sum_{k_1 > k_2 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{1}{k_j^{s_j}}, \qquad \sum_{j=1}^{n} \Re(s_j) > n, \quad n = 1, 2, \ldots, m, \tag{1.3}$$

have attracted increasing attention in recent years; see, for example, [3, 4, 5, 7, 8, 9, 10, 12, 14, 19]. The survey articles [6, 15, 22, 23, 25] provide an extensive list of references. In (1.3) the sum is over all positive integers $k_1, \ldots, k_m$ satisfying the indicated inequalities.

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Note that with positive integer arguments, $s_1 > 1$ is necessary and sufficient for convergence.

The problem of evaluating sums of the form (1.2) for integers $s > 1$, $t > 0$ seems to have been first proposed in a letter from Goldbach to Euler [17] in 1742. (See also [16, 18] and [1, page 253].) Among other results for (1.2), Euler proved that if $s-1$ and $t-1$ are positive integers, then the decomposition formula

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1}\binom{a+t-1}{t-1}\zeta(t+a,s-a) + \sum_{a=0}^{t-1}\binom{a+s-1}{s-1}\zeta(s+a,t-a) \tag{1.4}$$

holds. A combinatorial proof of Euler's decomposition formula (1.4) based on the simplex integral representations [3, 4, 5, 6, 7]

$$\begin{gathered}\zeta(s) = \int_{1>x_1>\cdots>x_s>0}\left(\prod_{i=1}^{s-1}\frac{dx_i}{x_i}\right)\frac{dx_s}{1-x_s},\\ \zeta(s,t) = \int_{1>x_1>\cdots>x_{s+t}>0}\left(\prod_{i=1}^{s-1}\frac{dx_i}{x_i}\right)\frac{dx_s}{1-x_s}\left(\prod_{i=s+1}^{s+t-1}\frac{dx_i}{x_i}\right)\frac{dx_{s+t}}{1-x_{s+t}},\end{gathered} \tag{1.5}$$

and the shuffle multiplication rule satisfied by such integrals is given in [4, (10)]. It is of course well known that (1.4) can also be proved algebraically by summing the partial fraction decomposition (see [21, page 48] and [20, Lemma 3.1])

$$\frac{1}{x^s(c-x)^t} = \sum_{a=0}^{s-1}\binom{a+t-1}{t-1}\frac{1}{x^{s-a}c^{t+a}} + \sum_{a=0}^{t-1}\binom{a+s-1}{s-1}\frac{1}{c^{s+a}(c-x)^{t-a}} \tag{1.6}$$

over appropriately chosen integers x and c. (See, e.g., [2].)

With the general goal of gaining a more complete understanding of the myriad relations satisfied by the multiple zeta functions (1.3) in mind, a q-analog of (1.3) was introduced in [11] as

$$\zeta[s_1,s_2,\ldots,s_m] := \sum_{k_1>k_2>\cdots>k_m>0}\prod_{j=1}^{m}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}, \tag{1.7}$$

where

$$[k]_q := \sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}, \quad 0 < q < 1. \tag{1.8}$$

Observe that we now have

$$\zeta(s_1,\ldots,s_m) = \lim_{q\to 1-}\zeta[s_1,\ldots,s_m], \tag{1.9}$$

so that (1.7) represents a generalization of (1.3). The paper [11] considers values of the multiple q-zeta functions (1.7) and establishes several infinite classes of relations satisfied by them. See also [13]. Here, we continue this general program of study by establishing a q-analog of Euler's decomposition formula (1.4).

2. Main result

Our q-analog of Euler's decomposition formula naturally requires only the $m = 1$ and $m = 2$ cases of (1.7); specifically the q-analogs of (1.1) and (1.2) given by

$$\zeta[s] = \sum_{n>0} \frac{q^{(s-1)n}}{[n]_q^s}, \qquad \zeta[s,t] = \sum_{n>k>0} \frac{q^{(s-1)n}q^{(t-1)k}}{[n]_q^s[k]_q^t}. \tag{2.1}$$

We also define, for convenience, the sum

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s]. \tag{2.2}$$

We can now state our main result.

Theorem 2.1. *If $s-1$ and $t-1$ are positive integers, then*

$$\begin{aligned}\zeta[s]\zeta[t] = &\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{t-1}\binom{t-1}{b}(1-q)^b\zeta[t+a,s-a-b] \\ &+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\zeta[s+a,t-a-b] \\ &-\sum_{j=1}^{\min(s,t)}\frac{(s+t-j-1)!}{(s-j)!(t-j)!}\cdot\frac{(1-q)^j}{(j-1)!}\varphi[s+t-j].\end{aligned} \tag{2.3}$$

Observe that the limiting case $q = 1$ of Theorem 2.1 reduces to Euler's decomposition formula (1.4).

3. A differential identity

Our proof of Theorem 2.1 relies on the following identity.

Lemma 3.1. *Let s and t be positive integers, and let x and y be nonzero real numbers. Then, for all real q such that $x+y+(q-1)xy \neq 0$,*

$$\begin{aligned}\frac{1}{x^s y^t} = &\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{t-1}\binom{t-1}{b}\frac{(1-q)^b(1+(q-1)y)^a(1+(q-1)x)^{t-1-b}}{x^{s-a-b}(x+y+(q-1)xy)^{t+a}} \\ &+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}\frac{(1-q)^b(1+(q-1)x)^a(1+(q-1)y)^{s-1-b}}{y^{t-a-b}(x+y+(q-1)xy)^{s+a}} \\ &-\sum_{j=1}^{\min(s,t)}\frac{(s+t-j-1)!}{(s-j)!(t-j)!}\cdot\frac{(1-q)^j}{(j-1)!}\cdot\frac{(1+(q-1)y)^{s-j}(1+(q-1)x)^{t-j}}{(x+y+(q-1)xy)^{s+t-j}}.\end{aligned} \tag{3.1}$$

Proof. Apply the partial differential operator

$$\frac{1}{(s-1)!}\left(-\frac{\partial}{\partial x}\right)^{s-1}\frac{1}{(t-1)!}\left(-\frac{\partial}{\partial y}\right)^{t-1} \tag{3.2}$$

to both sides of the identity

$$\frac{1}{xy}=\frac{1}{x+y+(q-1)xy}\left(\frac{1}{x}+\frac{1}{y}+q-1\right). \tag{3.3}$$

□

Observe that in the limit as $q \to 1$, Lemma 3.1 reduces to the identity

$$\frac{1}{x^s y^t}=\sum_{a=0}^{s-1}\binom{a+t-1}{t-1}\frac{1}{x^{s-a}(x+y)^{t+a}}+\sum_{a=0}^{t-1}\binom{a+s-1}{s-1}\frac{1}{(x+y)^{s+a}y^{t-a}}, \tag{3.4}$$

from which the partial fraction identity (1.6) (proved by induction in [20]) trivially follows.

4. Proof of Theorem 2.1

First, observe that if $s>1$ and $t>1$, then from (2.1),

$$\zeta[s]\zeta[t]=\sum_{n=1}^{\infty}\sum_{u+v=n}\frac{q^{(s-1)u}}{[u]_q^s}\cdot\frac{q^{(t-1)v}}{[v]_q^t}, \tag{4.1}$$

where the inner sum is over all positive integers u and v such that $u+v=n$. Next, apply Lemma 3.1 with $x=[u]_q$, $y=[v]_q$, noting that then

$$1+(q-1)x=q^u,\qquad 1+(q-1)y=q^v,\qquad x+y+(q-1)xy=[u+v]_q. \tag{4.2}$$

After interchanging the order of summation, there comes

$$\begin{aligned}\zeta[s]\zeta[t]=&\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{t-1}\binom{t-1}{b}(1-q)^b S[s,t,a,b]\\&+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b S[t,s,a,b]\\&-\sum_{j=1}^{\min(s,t)}\frac{(s+t-j-1)!}{(s-j)!(t-j)!}\cdot\frac{(1-q)^j}{(j-1)!}T[s,t,j],\end{aligned} \tag{4.3}$$

where

$$S[s,t,a,b] = \sum_{n=1}^{\infty}\sum_{u+v=n}\frac{q^{(s-1)u}q^{(t-1)v}q^{(t-1-b)u}q^{av}}{[u]_q^{s-a-b}[u+v]_q^{t+a}} = \sum_{n=1}^{\infty}\sum_{u+v=n}\frac{q^{(t+a-1)(u+v)}q^{(s-a-b-1)u}}{[u+v]_q^{t+a}[u]_q^{s-a-b}}$$

$$= \sum_{n=1}^{\infty}\frac{q^{(t+a-1)n}}{[n]_q^{t+a}}\sum_{u=1}^{n-1}\frac{q^{(s-a-b-1)u}}{[u]_q^{s-a-b}} = \zeta[t+a,s-a-b],$$

$$T[s,t,j] = \sum_{n=1}^{\infty}\sum_{u+v=n}\frac{q^{(s-1)u}q^{(t-1)v}q^{(t-j)u}q^{(s-j)v}}{[u+v]_q^{s+t-j}} = \sum_{n=1}^{\infty}\sum_{u+v=n}\frac{q^{(s+t-j-1)(u+v)}}{[u+v]_q^{s+t-j}} = \varphi[s+t-j]. \tag{4.4}$$

5. Final remarks

In [24], Zhao gives a much more complicated formula for the product $\zeta[s]\zeta[t]$. Zhao's formula is derived using the q-shuffle rule [6, 11] satisfied by the Jackson q-integral analogs of the representations (1.5). Of course from [11], we also have the very simple *q-stuffle* formula $\zeta[s]\zeta[t] = \zeta[s,t] + \zeta[t,s] + \zeta[s+t] + (1-q)\zeta[s+t-1]$ in which $s > 1$ and $t > 1$ need not be integers.

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