

A q -ANALOG OF EULER'S DECOMPOSITION FORMULA FOR THE DOUBLE ZETA FUNCTION

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The double zeta function was first studied by Euler in response to a letter from Goldbach in 1742. One of Euler's results for this function is a decomposition formula, which expresses the product of two values of the Riemann zeta function as a finite sum of double zeta values involving binomial coefficients. Here, we establish a q -analog of Euler's decomposition formula. More specifically, we show that Euler's decomposition formula can be extended to what might be referred to as a "double q -zeta function" in such a way that Euler's formula is recovered in the limit as q tends to 1.

1. Introduction

The Riemann zeta function is defined for $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (1.1)$$

Accordingly,

$$\zeta(s, t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \quad \Re(s) > 1, \quad \Re(s+t) > 2, \quad (1.2)$$

is known as the double zeta function. The sums (1.2), and more generally those of the form

$$\zeta(s_1, s_2, \dots, s_m) := \sum_{k_1 > k_2 > \dots > k_m > 0} \prod_{j=1}^m \frac{1}{k_j^{s_j}}, \quad \sum_{j=1}^m \Re(s_j) > n, \quad n = 1, 2, \dots, m, \quad (1.3)$$

have attracted increasing attention in recent years; see, for example, [3, 4, 5, 7, 8, 9, 10, 12, 14, 19]. The survey articles [6, 15, 22, 23, 25] provide an extensive list of references. In (1.3) the sum is over all positive integers k_1, \dots, k_m satisfying the indicated inequalities.

Note that with positive integer arguments, $s_1 > 1$ is necessary and sufficient for convergence.

The problem of evaluating sums of the form (1.2) for integers $s > 1, t > 0$ seems to have been first proposed in a letter from Goldbach to Euler [17] in 1742. (See also [16, 18] and [1, page 253].) Among other results for (1.2), Euler proved that if $s - 1$ and $t - 1$ are positive integers, then the decomposition formula

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \zeta(t+a, s-a) + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \zeta(s+a, t-a) \tag{1.4}$$

holds. A combinatorial proof of Euler's decomposition formula (1.4) based on the simplex integral representations [3, 4, 5, 6, 7]

$$\begin{aligned} \zeta(s) &= \int_{1 > x_1 > \dots > x_{s-1} > 0} \left(\prod_{i=1}^{s-1} \frac{dx_i}{x_i} \right) \frac{dx_s}{1-x_s}, \\ \zeta(s, t) &= \int_{1 > x_1 > \dots > x_{s+t} > 0} \left(\prod_{i=1}^{s-1} \frac{dx_i}{x_i} \right) \frac{dx_s}{1-x_s} \left(\prod_{i=s+1}^{s+t-1} \frac{dx_i}{x_i} \right) \frac{dx_{s+t}}{1-x_{s+t}}, \end{aligned} \tag{1.5}$$

and the shuffle multiplication rule satisfied by such integrals is given in [4, (10)]. It is of course well known that (1.4) can also be proved algebraically by summing the partial fraction decomposition (see [21, page 48] and [20, Lemma 3.1])

$$\frac{1}{x^s(c-x)^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a}c^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{c^{s+a}(c-x)^{t-a}} \tag{1.6}$$

over appropriately chosen integers x and c . (See, e.g., [2].)

With the general goal of gaining a more complete understanding of the myriad relations satisfied by the multiple zeta functions (1.3) in mind, a q -analog of (1.3) was introduced in [11] as

$$\zeta[s_1, s_2, \dots, s_m] := \sum_{k_1 > k_2 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}, \tag{1.7}$$

where

$$[k]_q := \sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}, \quad 0 < q < 1. \tag{1.8}$$

Observe that we now have

$$\zeta(s_1, \dots, s_m) = \lim_{q \rightarrow 1^-} \zeta[s_1, \dots, s_m], \tag{1.9}$$

so that (1.7) represents a generalization of (1.3). The paper [11] considers values of the multiple q -zeta functions (1.7) and establishes several infinite classes of relations satisfied by them. See also [13]. Here, we continue this general program of study by establishing a q -analog of Euler's decomposition formula (1.4).

2. Main result

Our q -analog of Euler's decomposition formula naturally requires only the $m = 1$ and $m = 2$ cases of (1.7); specifically the q -analogs of (1.1) and (1.2) given by

$$\zeta[s] = \sum_{n>0} \frac{q^{(s-1)n}}{[n]_q^s}, \quad \zeta[s, t] = \sum_{n>k>0} \frac{q^{(s-1)n} q^{(t-1)k}}{[n]_q^s [k]_q^t}. \quad (2.1)$$

We also define, for convenience, the sum

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s]. \quad (2.2)$$

We can now state our main result.

THEOREM 2.1. *If $s - 1$ and $t - 1$ are positive integers, then*

$$\begin{aligned} \zeta[s]\zeta[t] &= \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} (1-q)^b \zeta[t+a, s-a-b] \\ &+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b \zeta[s+a, t-a-b] \\ &- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \cdot \frac{(1-q)^j}{(j-1)!} \varphi[s+t-j]. \end{aligned} \quad (2.3)$$

Observe that the limiting case $q = 1$ of Theorem 2.1 reduces to Euler's decomposition formula (1.4).

3. A differential identity

Our proof of Theorem 2.1 relies on the following identity.

LEMMA 3.1. *Let s and t be positive integers, and let x and y be nonzero real numbers. Then, for all real q such that $x + y + (q-1)xy \neq 0$,*

$$\begin{aligned} \frac{1}{x^s y^t} &= \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} \frac{(1-q)^b (1+(q-1)y)^a (1+(q-1)x)^{t-1-b}}{x^{s-a-b} (x+y+(q-1)xy)^{t+a}} \\ &+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} \frac{(1-q)^b (1+(q-1)x)^a (1+(q-1)y)^{s-1-b}}{y^{t-a-b} (x+y+(q-1)xy)^{s+a}} \\ &- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \cdot \frac{(1-q)^j}{(j-1)!} \cdot \frac{(1+(q-1)y)^{s-j} (1+(q-1)x)^{t-j}}{(x+y+(q-1)xy)^{s+t-j}}. \end{aligned} \quad (3.1)$$

Proof. Apply the partial differential operator

$$\frac{1}{(s-1)!} \left(-\frac{\partial}{\partial x}\right)^{s-1} \frac{1}{(t-1)!} \left(-\frac{\partial}{\partial y}\right)^{t-1} \tag{3.2}$$

to both sides of the identity

$$\frac{1}{xy} = \frac{1}{x+y+(q-1)xy} \left(\frac{1}{x} + \frac{1}{y} + q-1\right). \tag{3.3}$$

□

Observe that in the limit as $q \rightarrow 1$, Lemma 3.1 reduces to the identity

$$\frac{1}{x^s y^t} = \sum_{a=0}^{s-1} \binom{a+t-1}{t-1} \frac{1}{x^{s-a}(x+y)^{t+a}} + \sum_{a=0}^{t-1} \binom{a+s-1}{s-1} \frac{1}{(x+y)^{s+a} y^{t-a}}, \tag{3.4}$$

from which the partial fraction identity (1.6) (proved by induction in [20]) trivially follows.

4. Proof of Theorem 2.1

First, observe that if $s > 1$ and $t > 1$, then from (2.1),

$$\zeta[s]\zeta[t] = \sum_{n=1}^{\infty} \sum_{u+v=n} \frac{q^{(s-1)u}}{[u]_q^s} \cdot \frac{q^{(t-1)v}}{[v]_q^t}, \tag{4.1}$$

where the inner sum is over all positive integers u and v such that $u + v = n$. Next, apply Lemma 3.1 with $x = [u]_q$, $y = [v]_q$, noting that then

$$1 + (q-1)x = q^u, \quad 1 + (q-1)y = q^v, \quad x + y + (q-1)xy = [u+v]_q. \tag{4.2}$$

After interchanging the order of summation, there comes

$$\begin{aligned} \zeta[s]\zeta[t] &= \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} (1-q)^b S[s, t, a, b] \\ &\quad + \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b S[t, s, a, b] \\ &\quad - \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \cdot \frac{(1-q)^j}{(j-1)!} T[s, t, j], \end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
 S[s, t, a, b] &= \sum_{n=1}^{\infty} \sum_{u+v=n} \frac{q^{(s-1)u} q^{(t-1)v} q^{(t-1-b)u} q^{av}}{[u]_q^{s-a-b} [u+v]_q^{t+a}} = \sum_{n=1}^{\infty} \sum_{u+v=n} \frac{q^{(t+a-1)(u+v)} q^{(s-a-b-1)u}}{[u+v]_q^{t+a} [u]_q^{s-a-b}} \\
 &= \sum_{n=1}^{\infty} \frac{q^{(t+a-1)n}}{[n]_q^{t+a}} \sum_{u=1}^{n-1} \frac{q^{(s-a-b-1)u}}{[u]_q^{s-a-b}} = \zeta[t+a, s-a-b], \\
 T[s, t, j] &= \sum_{n=1}^{\infty} \sum_{u+v=n} \frac{q^{(s-1)u} q^{(t-1)v} q^{(t-j)u} q^{(s-j)v}}{[u+v]_q^{s+t-j}} = \sum_{n=1}^{\infty} \sum_{u+v=n} \frac{q^{(s+t-j-1)(u+v)}}{[u+v]_q^{s+t-j}} = \varphi[s+t-j].
 \end{aligned} \tag{4.4}$$

5. Final remarks

In [24], Zhao gives a much more complicated formula for the product $\zeta[s]\zeta[t]$. Zhao's formula is derived using the q -shuffle rule [6, 11] satisfied by the Jackson q -integral analogs of the representations (1.5). Of course from [11], we also have the very simple q -shuffle formula $\zeta[s]\zeta[t] = \zeta[s, t] + \zeta[t, s] + \zeta[s+t] + (1-q)\zeta[s+t-1]$ in which $s > 1$ and $t > 1$ need not be integers.

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References

- [1] B. C. Berndt, *Ramanujan's Notebooks. Part I*, Springer, New York, 1985.
- [2] D. Borwein, J. M. Borwein, and R. Girgensohn, *Explicit evaluation of Euler sums*, Proc. Edinburgh Math. Soc. (2) **38** (1995), no. 2, 277–294.
- [3] J. M. Borwein, D. M. Bradley, and D. J. Broadhurst, *Evaluations of k -fold Euler/Zagier sums: a compendium of results for arbitrary k* , Electron. J. Combin. **4** (1997), no. 2, Research Paper 5, approx. 21, The Wilf Festschrift (Philadelphia, Pa, 1996).
- [4] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisoněk, *Combinatorial aspects of multiple zeta values*, Electron. J. Combin. **5** (1998), no. 1, Research Paper 38, 12.
- [5] ———, *Special values of multiple polylogarithms*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 907–941.
- [6] D. Bowman and D. M. Bradley, *Multiple polylogarithms: a brief survey*, q -Series with Applications to Combinatorics, Number Theory, and Physics (Urbana, Ill, 2000) (B. C. Berndt and K. Ono, eds.), Contemp. Math., vol. 291, American Mathematical Society, Rhode Island, 2001, pp. 71–92.
- [7] ———, *The algebra and combinatorics of shuffles and multiple zeta values*, J. Combin. Theory Ser. A **97** (2002), no. 1, 43–61.
- [8] ———, *Resolution of some open problems concerning multiple zeta evaluations of arbitrary depth*, Compositio Math. **139** (2003), no. 1, 85–100.
- [9] D. Bowman, D. M. Bradley, and J. H. Ryoo, *Some multi-set inclusions associated with shuffle convolutions and multiple zeta values*, European J. Combin. **24** (2003), no. 1, 121–127.
- [10] D. M. Bradley, *Duality for finite multiple harmonic q -series*, Discrete Math. **300** (2005), no. 1–3, 44–56.

- [11] ———, *Multiple q -zeta values*, J. Algebra **283** (2005), no. 2, 752–798.
- [12] ———, *Partition identities for the multiple zeta function*, Zeta Functions, Topology, and Quantum Physics (T. Aoki, S. Kanemitsu, M. Nakahara, and Y. Ohno, eds.), Springer Series: Developments in Mathematics, vol. 14, Springer, New York, 2005, pp. 19–29.
- [13] ———, *On the sum formula for multiple q -zeta values*, to appear in Rocky Mountain J. Math., <http://arxiv.org/abs/math.QA/0411274>.
- [14] D. J. Broadhurst and D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B **393** (1997), no. 3–4, 403–412.
- [15] P. Cartier, *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents* [*Polylogarithm functions, polyzeta numbers and pro-unipotent groups*], Astérisque **282** (2002), viii, 137–173, Séminaire Bourbaki, 53^eme année, 2000/2001, Exp. No. 885.
- [16] L. Euler, *Méditationes circa singulare serierum genus*, Novi Comm. Acad. Sci. Petropol. **20** (1775), 140–186, reprinted in “Opera Omnia,” ser. I, **15**, B. G. Teubner, Berlin (1927), 217–267.
- [17] ———, *Briefwechsel*, vol. 1, Birkhäuser, Basel, 1975.
- [18] L. Euler and C. Goldbach, *Briefwechsel 1729–1764*, Akademie, Berlin, 1965.
- [19] T. Q. T. Le and J. Murakami, *Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions*, Topology Appl. **62** (1995), no. 2, 193–206.
- [20] C. Markett, *Triple sums and the Riemann zeta function*, J. Number Theory **48** (1994), no. 2, 113–132.
- [21] N. Nielsen, *Die Gammafunktion. Band I. Handbuch der Theorie der Gammafunktion. Band II. Theorie des Integrallogarithmus und verwandter Transzendenten*, Chelsea, New York, 1965.
- [22] M. Waldschmidt, *Valeurs zêta multiples. Une introduction* [*Multiple zeta values: an introduction*], J. Théor. Nombres Bordeaux **12** (2000), no. 2, 581–595.
- [23] ———, *Multiple polylogarithms: an introduction*, Number Theory and Discrete Mathematics (Chandigarh, 2000), Trends Math., Birkhäuser, Basel, 2002, pp. 1–12.
- [24] J. Zhao, *q -multiple zeta functions and q -multiple polylogarithms*, <http://arxiv.org/abs/math.QA/0304448>, v2, May 2003.
- [25] V. V. Zudilin, *Algebraic relations for multiple zeta values*, Uspekhi Mat. Nauk **58** (2003), no. 1(349), 3–32 (Russian), translation in Russian Math. Surveys **58** (2003), no. 1, 1–29.

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