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Series Acceleration Formulas for Dirichlet Series with Periodic Coefficients

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Abstract. Series acceleration formulas are obtained for Dirichlet series with periodic coefficients. Special cases include Ramanujan's formula for the values of the Riemann zeta function at the odd positive integers exceeding two, and related formulas for values of Dirichlet L-series and the Lerch zeta function.

Key words: Dirichlet series, acceleration of series, L-series, Riemann zeta function, Lerch zeta function, Ramanujan

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1. Introduction

Let m be a positive integer and let g be a complex-valued function defined on the integers that is periodic with period m. In other words, $g : \mathbf{Z} \to \mathbf{C}$ has the property that $g(n+m) = g(n)$ for all integers n. Examples include the constant functions ($m = 1$) and the Dirichlet characters modulo $m > 1$, but it is not necessary in what follows to make any sort of multiplicativity assumptions on g. If $g : \mathbf{Z} \to \mathbf{C}$ has period m then g has mean value

$$M(g) = \frac{1}{m}\sum_{n=0}^{m-1} g(n), \tag{1.1}$$

and furthermore, since g is bounded ($\sup_{n\in\mathbf{Z}} |g(n)| = \max_{1\le n\le m} |g(n)| < \infty$), the Dirichlet series

$$L(s, g) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \tag{1.2}$$

converges absolutely in the half-plane $\{s \in \mathbf{C} : \Re(s) > 1\}$. An easy partial summation argument shows that the series (1.2) converges conditionally in the half-plane $\{s \in \mathbf{C} : \Re(s) > 0\}$ if and only if $M(g) = 0$.

Henceforth, we shall denote the abscissa of convergence of the Dirichlet series (1.2) by σ_g. That is,

$$\sigma_g := \inf\{\sigma \in \mathbf{R} : (1.2) \text{ converges for } \Re(s) > \sigma\}. \tag{1.3}$$

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Then the previous observations may be restated as asserting that $\sigma_g \le 1$ for all periodic $g : \mathbf{Z} \to \mathbf{C}$, and $\sigma_g \le 0$ if and only if $M(g) = 0$. Of course if $g : \mathbf{Z} \to \mathbf{C}$ is periodic and $M(g) = 0$, then we actually must have $\sigma_g = 0$ except in the trivial case when g vanishes identically and $\sigma_g = -\infty$.

As part of a general program aimed at developing methods for calculating important number-theoretical constants to high precision, we consider here the problem of replacing values of the series (1.2) by equivalent expressions with improved rate of convergence. If we succeed in finding such an expression for a specific value of s, we say we have obtained a *series acceleration formula* for $L(s, g)$. Theorems 1 and 2 below provide series acceleration formulas for $L(s, g)$ when $g : \mathbf{Z} \to \mathbf{C}$ is periodic and either odd or even and s is a positive integer. Special cases include Ramanujan's beautiful reciprocity formula for the values $\zeta(2q+1)$ of the Riemann zeta function (where q is a positive integer), analogous formulas for values of Dirichlet L-series, and other related results that have appeared in the literature.

We make no attempt here to give a rigorous definition of the concept of a series acceleration formula. However, the following remarks should give a reasonable indication of what we mean. All our results have the form $\sum a_n = C + S$, where C is a "closed form" expression and S is a finite sum of series (equivalently, a single series) of the form $\sum b_n$, in which b_n is an elementary function of n and $\limsup_{n\to\infty} |b_n/a_n|^{1/n} < 1$. Of course, "closed form" depends on what sort of objects one is prepared to accept as fundamental. Let us agree to accept values of the elementary functions at the integers as closed form. Then (see Proposition 1) $L(q, g)$ is closed form when q is a positive integer such that g and q are of the same parity, and in particular, the numbers $\zeta(2q)$ are closed form. For even g, the second formula in Theorem 1 expresses $L(2q+1, g)$ in terms of the aforementioned closed form values, rapidly convergent series, and $\zeta(2q+1)$. Since we do not regard $\zeta(2q+1)$ as closed form, a series acceleration formula for $\zeta(2q+1)$ is needed in order to achieve a series acceleration formula for $L(2q+1, g)$. Fortunately, Ramanujan's formula for $\zeta(2q+1)$ (see Corollary 3) serves quite adequately as a series acceleration formula, and thus Theorems 1 and 2 do indeed provide legitimate series acceleration formulas for $L(s, g)$, as claimed.

2. Main result

Theorem 1. *Let m and q be positive integers, $\omega = \exp(2\pi i/m)$, $g : \mathbf{Z} \to \mathbf{C}$ periodic of period m, $M(g)$ as in* (1.1) *and $L(s, g)$ as in* (1.2). *If α and β are positive real numbers satisfying $\alpha\beta = \pi^2$, then*

$$\alpha^{-q+1/2}\left\{\frac{1}{2}L(2q, g) + \sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2n\alpha}-1}\right\}$$
$$= (-1)^q \beta^{-q+1/2} i m^{-1} \sum_{k=1}^{m-1} g(k) \sum_{n=1}^{\infty} \frac{n^{-2q}}{e^{2n\beta/m}\omega^k - 1}$$
$$+ \sum_{j=0}^{q} (-1)^{j+1} \alpha^{j-1/2-q} \beta^{-j} \zeta(2j) L(2q-2j+1, g)$$

if g is odd, and

$$\alpha^{-q}\left\{\tfrac{1}{2}L(2q+1,g)+\sum_{n=1}^{\infty}\frac{n^{-2q-1}g(n)}{e^{2n\alpha}-1}\right\}$$
$$=(-\beta)^{-q}\left\{\tfrac{1}{2}M(g)\zeta(2q+1)+\frac{1}{m}\sum_{k=0}^{m-1}g(k)\sum_{n=1}^{\infty}\frac{n^{-2q-1}}{e^{2n\beta/m}\omega^k-1}\right\}$$
$$+\sum_{j=0}^{q+1}(-1)^{j+1}\alpha^{j-q-1}\beta^{-j}\zeta(2j)L(2q-2j+2,g)$$

if g is even. If $M(g)=0$, the latter formula is also valid when $q=0$.

There is an equivalent version of Theorem 1 in which the terms involving complex mth roots of unity are paired so as to yield a real-valued expression:

Theorem 2. *Let m and q be positive integers, $g:\mathbf{Z}\to\mathbf{C}$ periodic of period m, $M(g)$ as in (1.1) and $L(s,g)$ as in (1.2). If α and β are positive real numbers satisfying $\alpha\beta=\pi^2$, then*

$$\alpha^{-q+1/2}\left\{\tfrac{1}{2}L(2q,g)+\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2n\alpha}-1}\right\}$$
$$=\tfrac{1}{2}(-1)^q\beta^{-q+1/2}m^{-1}\sum_{k=1}^{m-1}g(k)\sin(2\pi k/m)\sum_{n=1}^{\infty}\frac{n^{-2q}}{\cosh(2n\beta/m)-\cos(2\pi k/m)}$$
$$+\sum_{j=0}^{q}(-1)^{j+1}\alpha^{j-1/2-q}\beta^{-j}\zeta(2j)L(2q-2j+1,g)$$

if g is odd, and

$$\alpha^{-q}\left\{\tfrac{1}{2}L(2q+1,g)+\sum_{n=1}^{\infty}\frac{n^{-2q-1}g(n)}{e^{2n\alpha}-1}\right\}$$
$$=\tfrac{1}{2}(-\beta)^{-q}m^{-1}\sum_{k=0}^{m-1}g(k)\sum_{n=1}^{\infty}\left(\frac{\cos(2\pi k/m)-\exp(-2n\beta/m)}{\cosh(2n\beta/m)-\cos(2\pi k/m)}\right)n^{-2q-1}$$
$$+\tfrac{1}{2}(-\beta)^{-q}M(g)\zeta(2q+1)+\sum_{j=0}^{q+1}(-1)^{j+1}\alpha^{j-q-1}\beta^{-j}\zeta(2j)L(2q-2j+2,g)$$

if g is even. If $M(g)=0$, the latter formula is also valid when $q=0$.

Our proof of Theorems 1 and 2 is outlined in Section 4. It should be noted that Ramanujan had a generalization of Theorem 1 in which g is replaced by an entire function satisfying suitable growth conditions. Ramanujan's generalization is proved using contour integration in [6, pp. 429–430]. Although our results are less general than Ramanujan's, our method of proof is somewhat more elementary than previous approaches, and therefore may be of some interest.

As we remarked in the Introduction, it should be noted that for these results to be of use as series acceleration formulas, it is necessary to have a series acceleration formula for $\zeta(2q+1)$ in the case when g is even and $M(g) \neq 0$, and a closed-form evaluation of $L(s, g)$ when s is a non-negative integer of the same parity as g. For $\zeta(2q+1)$, we may use Ramanujan's formula—see Corollary 3 below. For $L(s, g)$, we first recall that if $g = \chi$ is a Dirichlet character and s is a positive integer such that $(-1)^s\chi(-1) = 1$, then the corresponding Dirichlet L-series $L(s, \chi)$ has a closed-form evaluation in terms of the Gauss sum

$$G(\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}$$

and the so-called *generalized Bernoulli numbers* $B_{n,\chi}$, defined for non-negative integers n by the formula

$$\sum_{k=1}^{m} \chi(k) \frac{te^{kt}}{e^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

This well-known result is typically derived (see e.g. [16]) as a consequence of the functional equation relating $L(s, \chi)$ to $L(1-s, \bar{\chi})$, but actually one does not need the functional equation to accomplish this, and in fact, one does not even need the multiplicativity property of the Dirichlet characters.

As customary, define the Bernoulli polynomials $B_n(x)$ by their exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{2.1}$$

and the Bernoulli numbers by $B_n = B_n(0)$ for $0 \le n \in \mathbf{Z}$. The following closed-form evaluation for $L(s, g)$ makes no multiplicativity assumption on the periodic arithmetical function g.

Proposition 1 ([8]). *Let m be a positive integer, $g : \mathbf{Z} \to \mathbf{C}$ periodic with period m and let $L(s, g)$ be as in* (1.2). *If q is a non-negative integer such that g and q are both odd or both even, then*

$$L(q, g) = -\frac{1}{2} \cdot \frac{(2\pi i)^q}{q!} \sum_{k=0}^{m-1} \hat{g}(k) B_q(k/m),$$

where $\hat{g}(k)$ is the kth discrete Fourier coefficient of g defined by

$$\hat{g}(k) := \frac{1}{m} \sum_{j=0}^{m-1} g(j)\, e^{-2\pi i jk/m}.$$

Proof: See Eqs. (6.23) and (6.25) of [8] for the case when q is a positive integer. For the case $q = 0$, we must show that if g is even, then

$$L(0, g) = -\frac{1}{2}\sum_{k=0}^{m-1}\hat{g}(k) = -\frac{1}{2m}\sum_{k=0}^{m-1}\sum_{j=0}^{m-1} g(j)e^{-2\pi ijk/m} = -\frac{1}{2m}\sum_{j=0}^{m-1} g(j)\sum_{k=0}^{m-1} e^{-2\pi ijk/m}$$
$$= -\tfrac{1}{2}g(0).$$

But this one can easily establish by meromorphically extending the definition of $L(s, g)$ to the half-plane $\{s \in \mathbf{C} : \Re(s) > -1\}$. □

The equivalent "real" version of Proposition 1 is stated below for convenience.

Corollary 1. *Let m be a positive integer, $g : \mathbf{Z} \to \mathbf{C}$ periodic with period m and let $L(s, g)$ be as in* (1.2)*. Then for all non-negative integers q,*

$$L(2q, g) = \frac{(-1)^{q+1}(2\pi)^{2q}}{(2q)!} \cdot \frac{1}{2m}\sum_{k=0}^{m-1} B_{2q}(k/m)\sum_{j=0}^{m-1} g(j)\cos(2\pi jk/m)$$

if g is even, and

$$L(2q+1, g) = \frac{(-1)^{q+1}(2\pi)^{2q+1}}{(2q+1)!} \cdot \frac{1}{2m}\sum_{k=0}^{m-1} B_{2q+1}(k/m)\sum_{j=0}^{m-1} g(j)\sin(2\pi jk/m)$$

if g is odd.

In particular, we have the following well-known evaluations which are needed in the sequel:

Corollary 2. *For $\Re(s) > 0$, let $L(s) := \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-s}$, and let $\zeta(s)$ denote the Riemann zeta function. If n is a non-negative integer, then*

$$\zeta(2n) = -\frac{1}{2} \cdot \frac{(2\pi i)^{2n} B_{2n}}{(2n)!} \tag{2.2}$$

and

$$L(2n+1) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1} \frac{(-1)^n E_{2n}}{(2n)!}, \tag{2.3}$$

where

$$\frac{1}{\cosh(t)} = \sum_{k=0}^{\infty} \frac{E_k}{k!} t^k, \quad |t| < \tfrac{1}{2}\pi,$$

generates the Euler numbers E_k for non-negative integers k.

3. Consequences

Corollary 3 (*Ramanujan's formula for* $\zeta(2q+1)$)**.** *Let q be a positive integer, and let α and β be positive real numbers with $\alpha\beta = \pi^2$. Then*

$$\alpha^{-q}\left\{\frac{1}{2}\zeta(2q+1) + \sum_{n=1}^{\infty}\frac{n^{-2q-1}}{e^{2n\alpha}-1}\right\} = (-\beta)^{-q}\left\{\frac{1}{2}\zeta(2q+1) + \sum_{n=1}^{\infty}\frac{n^{-2q-1}}{e^{2n\beta}-1}\right\} + 2^{2q}\sum_{k=0}^{q+1}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{2q+2-2k}}{(2q+2-2k)!}\,\alpha^{q+1-k}\beta^{k}. \tag{3.1}$$

Proof: Let $m = 1$ in Theorem 1 and let $g(0) = 1$, noting that periodicity implies $g \equiv 1$ is constant. The stated formula now follows after appropriately substituting (2.2) and performing standard algebraic manipulations. □

Remarks. Corollary 3 appears in Ramanujan's notebooks [18, vol. I, p. 259, no. 15; vol. II, p. 177, no. 21], but he did not publish a proof. Although Lerch [15] proved the special case $\alpha = \beta = \pi$, Grosswald's paper [11] was responsible for generating much of the subsequent interest in (3.1). Guinand [12] showed how Corollary 3 arises from the modular transformation $V(z) = -1/z$ of the function

$$f(z, -2n) = \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{-2\pi ikz}-1}, \quad \Im(z) > 0,$$

and letting $z = i$. Additional formulas for $\zeta(2q+1)$ and cognate results for certain other Dirichlet series may be obtained by applying other transformations. See [7] for a comprehensive account with extensive references to the literature. Many further references can be found in [5, p. 276]. More recently, Ramanujan's formula for $\zeta(2q+1)$ has been applied [14] in studying the variance of the random variable X_q representing the number of internal nodes of a binary trie built from q data.

Corollary 4. *For $\Re(s) > 0$, let $L(s) := \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}$. Let q be a positive integer, and let α and β be positive real numbers with $\alpha\beta = \pi^2$. Then*

$$\alpha^{-q+1/2}\left\{\frac{1}{2}L(2q) + \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^{-2q}}{e^{(4n+2)\alpha}-1}\right\} = \frac{1}{4}(-1)^q\beta^{-q+1/2}\sum_{n=1}^{\infty}\frac{n^{-2q}}{\cosh(n\beta/2)} + 2^{2q-3}\sum_{k=0}^{q}(-1)^k 2^{-4k}\frac{E_{2k}}{(2k)!}\frac{B_{2q-2k}}{(2q-2k)!}\alpha^{q-k}\beta^{k+1/2}.$$

Proof: Put $m = 4$ in Theorem 2 and let $g(2n+1) = (-1)^n$ and $g(2n) = 0$ for $n \in \mathbf{Z}$. The stated formula now follows after appropriately substituting (2.2) and (2.3) and performing standard algebraic manipulations. □

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Remarks. Corollary 4 corrects the misprints in Proposition 3.5 of [7, p. 169]. As noted by Berndt [7], the result was known to Ramanujan [18, Vol. I, p. 274; Vol. II, pp. 177 and 178], but the first published proof is due to Chowla [9]. Note that $L(s)$ is the Dirichlet L-series corresponding to the primitive non-principal Dirichlet character modulo 4. In a similar vein, formulas for general Dirichlet L-functions have been given by Katayama [13] and Berndt [3]. Of course, since all Dirichlet L-series have periodic coefficients, our Theorems 1 and 2 can be specialized to give "Ramanujan formulas" for general Dirichlet L-functions as well.

For purposes of maximizing the rate of convergence of the most slowly convergent series in Theorems 1 and 2, the optimal choice of α and β is $\alpha = \pi/\sqrt{m}$, $\beta = \pi\sqrt{m}$, respectively. If this choice is made in Corollary 4 with $\sqrt{m} = q = 2$, we recover Ramanujan's formula [17, p. 43] for Catalan's constant:

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{5}{48}\pi^2 - 2\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^{-2}}{e^{(2n+1)\pi}-1} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2\cosh(\pi n)}. \tag{3.2}$$

Corollary 5 (*Theorem 3.3 of* [7])**.** *Let q be a non-negative integer and let α, β and r be positive real numbers with $0 < r < 1$ and $\alpha\beta = \pi^2$. Then*

$$\begin{aligned} &\alpha^{-q}\left\{\frac{1}{2}\sum_{n=1}^{\infty} n^{-2q-1}\cos(2\pi nr) + \sum_{n=1}^{\infty} \frac{n^{-2q-1}\cos(2\pi nr)}{e^{2n\alpha}-1}\right\} \\ &\quad = (-\beta)^{-q}\left\{\frac{1}{2}\sum_{n=1}^{\infty} n^{-2q-1}e^{-2n\beta r} + \sum_{n=1}^{\infty} \frac{n^{-2q-1}\cosh(2n\beta r)}{e^{2n\beta}-1}\right\} \\ &\quad\quad - 2^{2q}\sum_{k=0}^{q+1}(-1)^k \frac{B_{2k}(r)}{(2k)!}\frac{B_{2q+2-2k}}{(2q+2-2k)!}\,\alpha^{q+1-k}\beta^k; \end{aligned} \tag{3.3}$$

and if q is a positive integer, then

$$\begin{aligned} &\alpha^{-q+1/2}\left\{\frac{1}{2}\sum_{n=1}^{\infty} n^{-2q}\sin(2\pi nr) + \sum_{n=1}^{\infty} \frac{n^{-2q}\sinh(2\pi nr)}{e^{2n\alpha}-1}\right\} \\ &\quad = (-1)^q\beta^{-q+1/2}\left\{\frac{1}{2}\sum_{n=1}^{\infty} n^{-2q}e^{-2n\beta r} + \sum_{n=1}^{\infty} \frac{n^{-2q}\sinh(2n\beta r)}{e^{2n\beta}-1}\right\} \\ &\quad\quad - 2^{2q-1}\sum_{k=0}^{q}(-1)^k \frac{B_{2k+1}(r)}{(2k+1)!}\frac{B_{2q-2k}}{(2q-2k)!}\,\alpha^{q-k}\beta^{k+1/2}. \end{aligned} \tag{3.4}$$

Proof: It suffices to prove the given formulas in the case when $0 < r < 1$ and r is rational, as the result for real r with $0 < r < 1$ then follows by taking limits. We shall prove only (3.3), as the proof of (3.4) is almost identical. Suppose u and m are integers satisfying $0 < u < m$. Let $r = u/m$, and let $g : \mathbf{Z} \to \mathbf{C}$ be defined by $g(n) = \cos(2\pi nr)$ for all integers n. Then g

is even and periodic with period m. Rewrite Theorem 1 in the form

$$\tfrac{1}{2}\alpha^{-q}\sum_{n=1}^{\infty} n^{-2q-1}\coth(n\alpha) = \tfrac{1}{2}(-\beta)^{-q}\sum_{n=1}^{\infty} n^{-2q-1}\coth((n\beta+\pi ik)/m) - \sum_{k=0}^{q+1}(-1)^{q+1-k}L(2k,g)\zeta(2q-2k+2)\alpha^{-k}\beta^{k-q-1}.$$

From Proposition 1, we have

$$L(2k,g) = -\frac{1}{2}\cdot\frac{(2\pi i)^{2k}B_{2k}(r)}{(2k)!},\quad 0\le k\in\mathbf{Z}.$$

Using also (2.2), we see that it now suffices (with $\lambda = n\beta$) to prove that for all $\lambda > 0$,

$$m^{-1}\sum_{k=0}^{m-1}\cos(2\pi kr)\coth((\lambda+\pi ik)/m) = e^{-2\lambda r} + \frac{2\cosh(2\lambda r)}{e^{2\lambda}-1} = \frac{\cosh((1-2r)\lambda)}{\sinh(\lambda)}.$$

But

$$\begin{aligned}
&m^{-1}\sum_{k=0}^{m-1}\cos(2\pi kr)\coth((\lambda+\pi ik)/m)\\
&= \frac{1}{2m}\sum_{k=0}^{m-1}(e^{2\pi iku/m}+e^{-2\pi iku/m})\frac{1+e^{-2(\lambda+\pi ik)/m}}{1-e^{-2(\lambda+\pi ik)/m}}\\
&= \frac{1}{2m}\sum_{k=0}^{m-1}(e^{2\pi iku/m}+e^{-2\pi iku/m})\big(1+e^{-2(\lambda+\pi ik)/m}\big)\sum_{j=0}^{\infty}e^{-2(\lambda+\pi ik)j/m}\\
&= \frac{1}{2m}\sum_{j=0}^{\infty}e^{-2\lambda j/m}\sum_{k=0}^{m-1}\big(e^{2\pi ik(u-j)/m}+e^{-2\pi ik(u+j)/m}\big)\\
&\quad + \frac{1}{2m}\sum_{j=1}^{\infty}e^{-2\lambda j/m}\sum_{k=0}^{m-1}\big(e^{2\pi ik(u-j)/m}+e^{-2\pi ik(u+j)/m}\big)\\
&= \sum_{k=0}^{\infty}\big(e^{-2\lambda(u+km)/m}+e^{-2\lambda(m-u+km)/m}\big)\\
&= \frac{e^{-2\lambda r}}{1-e^{-2\lambda}}+\frac{e^{-2\lambda(1-r)}}{1-e^{-2\lambda}} = \frac{e^{-(1-2r)\lambda}+e^{(1-2r)\lambda}}{e^{\lambda}-e^{-\lambda}}\\
&= \frac{\cosh((1-2r)\lambda)}{\sinh(\lambda)}.
\end{aligned}$$

This completes the proof of (3.3). The proof of (3.4) proceeds mutatis mutandis with $g(n) = \sin(2\pi nr)$ replacing $g(n) = \cos(2\pi nr)$. □

Remarks. In (3.4), we have corrected the misprints in the corresponding formula (3.12) of [7, p. 167]. Berndt [4] has given a generalization of Corollary 5 to periodic sequences.

In [7], Berndt deduces several interesting results by specializing (3.3) and (3.4) in various ways. Here, we confine ourselves to remarking that setting $r = 1/4$ gives Corollary 4, and letting r tend to zero gives Euler's formula (2.2) and Ramanujan's formula (3.1) again.

4. Proof of main result

Although the results of Section 3 have been previously derived as consequences of quite general modular transformation formulas [7], we feel that it may nevertheless be of interest to give a proof of our main result using elementary methods analogous to Ramanujan's [17] proof of (3.2). The main idea is to employ the partial fraction expansion of the hyperbolic cotangent in a non-trivial manner. The proof has been broken down into a sequence of relatively straightforward lemmata, the last of which gives our main result after making a trivial substitution.

Lemma 1. *Let $f : \mathbf{Z}^+ \to \mathbf{C}$ and suppose that the associated Dirichlet series $F(s) := \sum_{n=1}^{\infty} n^{-s} f(n)$ has abscissa of convergence $\sigma_f < \infty$. If $\Re(s) > \sigma_f - 1$, $g : \mathbf{Z} \to \mathbf{C}$ and x is any positive real number, then*

$$\tfrac{1}{2}\pi x F(s+1) + \pi x \sum_{k=1}^{\infty} \frac{k^{-s-1} f(k)}{e^{2\pi k/x} - 1} = \tfrac{1}{2} x^2 F(s+2) + x^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{k^{-s} f(k)}{k^2 + n^2 x^2}.$$

Proof: Recalling the partial fraction expansion of the hyperbolic cotangent [1, p. 259, 6.3.13], we find that

$$\begin{aligned}
\tfrac{1}{2}\pi x F(s+1) + \pi x \sum_{k=1}^{\infty} \frac{k^{-s-1} f(k)}{e^{2\pi k/x} - 1} &= \tfrac{1}{2}\pi x \sum_{k=1}^{\infty} k^{-s-1} f(k) \left(1 + \frac{2}{e^{2\pi k/x} - 1}\right) \\
&= \tfrac{1}{2}\pi x \sum_{k=1}^{\infty} k^{-s-1} f(k) \coth(\pi k/x) \\
&= \sum_{k=1}^{\infty} k^{-s} f(k) \left(\tfrac{1}{2} x^2 k^{-2} + \sum_{n=1}^{\infty} \frac{x^2}{k^2 + n^2 x^2}\right) \\
&= \tfrac{1}{2} x^2 F(s+2) + \sum_{k=1}^{\infty} k^{-s} f(k) \sum_{n=1}^{\infty} \frac{x^2}{k^2 + n^2 x^2} \\
&= \tfrac{1}{2} x^2 F(s+2) + x^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{k^{-s} f(k)}{k^2 + n^2 x^2}.
\end{aligned}$$

The interchange of summation in the final step can be justified by the Fubini-Tonelli theorem with counting measure [10, p. 67]. The double sum is absolutely convergent because if $0 < \varepsilon < \min(1, \Re(s) + 1 - \sigma_f)$, then concavity of the logarithm implies that

$$\frac{1}{k^2 + n^2 x^2} \le \frac{2}{(1-\varepsilon)k^2 + (1+\varepsilon)n^2 x^2} \le \frac{1}{k^{1-\varepsilon}(nx)^{1+\varepsilon}},$$

and so

$$\sum_{n,k=1}^{\infty}\left|\frac{k^{-s}f(k)}{k^2+n^2x^2}\right| \le \sum_{n=1}^{\infty}\frac{1}{|nx|^{1+\varepsilon}}\sum_{k=1}^{\infty}\left|\frac{f(k)}{k^{s+1-\varepsilon}}\right| < \infty,$$

since $\Re(s)+1-\varepsilon > \sigma_f$. □

Lemma 2. *Let $f : \mathbf{Z}^+ \to \mathbf{C}$ and suppose that the associated Dirichlet series $F(s) := \sum_{n=1}^{\infty} n^{-s} f(n)$ has abscissa of convergence $\sigma_f < \infty$. For $\Re(s) > \sigma_f - 2$ and y real, define*

$$T_f(s, y) := \sum_{k=1}^{\infty}\frac{k^{-s}f(k)}{k^2+y^2}. \tag{4.1}$$

If q is a non-negative integer, $\Re(s) > \sigma_f + 2q - 2$ and $y \neq 0$, then

$$T_f(s, y) = (-1)^q y^{-2q} T_f(s-2q, y) + \sum_{j=1}^{q}(-1)^{j+1} y^{-2j} F(s-2j+2).$$

Proof: First, note that $T_f(s, 0) = F(s+2)$, so the series (4.1) defining $T_f(s, y)$ converges if $\Re(s) > \sigma_f - 2$. Next, observe that the formula clearly holds if $q = 0$, so we may assume that q is a positive integer. If $\Re(s) > \sigma_f$ and $y \neq 0$, then

$$T_f(s, y) = y^{-2}\sum_{k=1}^{\infty}(k^{-2}-(k^2+y^2)^{-1})k^{-s+2}f(k) = y^{-2}F(s) - y^{-2}T_f(s-2, y). \tag{4.2}$$

The stated formula now follows by iterating (4.2) or alternatively by replacing s by $s-2j+2$, multiplying both sides by $(-1)^{j+1}y^{-2j+2}$ and telescoping the sum on j from 1 to q. □

Lemma 3. *Let $f : \mathbf{Z}^+ \to \mathbf{C}$ and suppose that the associated Dirichlet series $F(s) := \sum_{n=1}^{\infty} n^{-s} f(n)$ has abscissa of convergence $\sigma_f < \infty$. Let q be a positive integer, x a positive real number, and T_f as in (4.1). Then for $\Re(s) > \sigma_f + 2q - 2$, we have*

$$\tfrac{1}{2}\pi x F(s+1) + \pi x \sum_{n=1}^{\infty}\frac{n^{-s-1}f(n)}{e^{2\pi n/x}-1}$$
$$= \sum_{j=0}^{q}(-1)^{j+1}x^{-2j+2}\zeta(2j)F(s-2j+2) + (-1)^q x^{-2q+2}\sum_{n=1}^{\infty}n^{-2q}T_f(s-2q, nx).$$

Proof: By Lemmas 1 and 2, we have

$$\tfrac{1}{2}\pi x F(s+1) + \pi x \sum_{n=1}^{\infty}\frac{n^{-s-1}f(n)}{e^{2\pi n/x}-1}$$
$$= \tfrac{1}{2}x^2 F(s+2) + x^2\sum_{n=1}^{\infty}T_f(s, nx)$$

$$= \tfrac{1}{2}x^2 F(s+2) + x^2 \sum_{n=1}^{\infty} \Bigg\{ (-1)^q n^{-2q} x^{-2q}\, T_f(s-2q, nx) + \sum_{j=1}^{q} (-1)^{j+1} n^{-2j} x^{-2j} F(s-2j+2) \Bigg\}$$

$$= \tfrac{1}{2}x^2 F(s+2) + \sum_{j=1}^{q} (-1)^{j+1} \zeta(2j) x^{-2j+2} F(s-2j+2) + (-1)^q x^{-2q+2} \sum_{n=1}^{\infty} n^{-2q}\, T_f(s-2q, nx).$$

Since $\zeta(0) = -1/2$, the sum on j can be extended to absorb the term $\frac{1}{2}x^2 F(s+2)$, and the stated formula follows. □

Lemma 4. *Let m be a positive integer, y a positive real number, and $g : \mathbf{Z} \to \mathbf{C}$ periodic of period m. Then*

$$P.V. \sum_{0 \neq k \in \mathbf{Z}} \frac{g(k)}{y + ik} = -y^{-1} g(0) + \pi m^{-1} \sum_{k=0}^{m-1} g(k) \coth(\pi(y+ik)/m),$$

where "P.V." denotes the principal value, i.e. the symmetric limit

$$\lim_{N \to \infty} \sum_{0 < |k| < N} \frac{g(k)}{y + ik}.$$

Proof: Since g has period m, we have

$$P.V. \sum_{0 \neq k \in \mathbf{Z}} \frac{g(k)}{y + ik} = -y^{-1} g(0) + \sum_{k=0}^{m-1} P.V. \sum_{r \in \mathbf{Z}} \frac{g(k+mr)}{y + i(k+mr)}$$

$$= -y^{-1} g(0) + \sum_{k=0}^{m-1} g(k)\, P.V. \sum_{r \in \mathbf{Z}} \frac{1/m}{(y+ik)/m + ir}$$

$$= -y^{-1} g(0) + \pi m^{-1} \sum_{k=0}^{m-1} g(k) \coth(\pi(y+ik)/m),$$

as stated. □

Lemma 5. *Let m be a positive integer, $\omega := \exp(2\pi i/m)$, y a positive real number, $g : \mathbf{Z} \to \mathbf{C}$ odd and periodic of period m, $L(s, g)$ as in* (1.2), *and σ_g as in* (1.3). *As in* (4.1), *for $\Re(s) > \sigma_g - 2$, define*

$$T_g(s, y) := \sum_{k=1}^{\infty} \frac{k^{-s} g(k)}{k^2 + y^2}. \tag{4.3}$$

Then

$$T_g(-1, y) = \sum_{k=1}^{\infty} \frac{kg(k)}{k^2 + y^2} = \tfrac{1}{2}\pi i m^{-1} \sum_{k=1}^{m-1} g(k) \coth(\pi(y + ik)/m) \tag{4.4}$$

$$= \frac{\pi i}{m} \sum_{k=1}^{m-1} \frac{g(k)}{e^{2\pi y/m}\omega^k - 1} \tag{4.5}$$

$$= \frac{\pi}{2m} \sum_{k=1}^{m-1} \frac{g(k)\sin(2\pi k/m)}{\cosh(2\pi y/m) - \cos(2\pi k/m)}. \tag{4.6}$$

Proof: We may as well assume g does not vanish identically, for otherwise the result is completely trivial. Since g has period m, $g(0) = g(m) = g(-m)$. Thus, as g is odd, $g(0) = \frac{1}{2}(g(m) + g(-m)) = 0$. Furthermore,

$$2mM(g) = \sum_{k=0}^{m-1} (g(k) + g(m - k)) = \sum_{k=0}^{m-1} (g(k) + g(-k)) = 0.$$

Therefore, by the remarks in the second paragraph of the Introduction, $\sigma_g = 0$, and the series defining $T_g(-1, y)$ converges. But since g is odd,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{kg(k)}{k^2 + y^2} &= \frac{i}{2} \sum_{k=1}^{\infty} \left(\frac{1}{y + ik} - \frac{1}{y - ik} \right) g(k) \\ &= \frac{i}{2} \sum_{k=1}^{\infty} \left(\frac{g(k)}{y + ik} + \frac{g(-k)}{y - ik} \right) \\ &= \frac{i}{2} P.V. \sum_{0 \neq k \in \mathbf{Z}} \frac{g(k)}{y + ik} \\ &= \tfrac{1}{2}\pi i m^{-1} \sum_{k=1}^{m-1} g(k) \coth(\pi(y + ik)/m)\end{aligned}$$

by Lemma 4 and the fact that $g(0) = 0$. This establishes (4.4).

Next, since $g(0) = M(g) = 0$,

$$\begin{aligned}\tfrac{1}{2}\pi i m^{-1} \sum_{k=1}^{m-1} g(k) \coth(\pi(y + ik)/m) &= \frac{\pi i}{2m} \sum_{k=1}^{m-1} g(k) \left(1 + \frac{2}{e^{2(y+ik)\pi/m} - 1} \right) \\ &= \tfrac{1}{2}\pi i M(g) + \frac{\pi i}{m} \sum_{k=1}^{m-1} \frac{g(k)}{e^{2\pi(y+ik)/m} - 1} \\ &= \frac{\pi i}{m} \sum_{k=1}^{m-1} \frac{g(k)}{e^{2\pi y/m}\omega^k - 1},\end{aligned}$$

which proves (4.5).

Finally, define $E : \mathbf{C} \to \mathbf{C}$ by $E(z) := \exp(2\pi z/m)$. Then (4.5) can be restated as

$$\begin{aligned} T_g(-1, y) &= \frac{\pi i}{m} \sum_{k=1}^{m-1} \frac{g(k)}{E(y+ik)-1} \\ &= \frac{\pi i}{2m} \sum_{k=1}^{m-1} \left(\frac{g(k)}{E(y+ik)-1} + \frac{g(m-k)}{E(y+i(m-k))-1} \right) \\ &= \frac{\pi i}{2m} \sum_{k=1}^{m-1} \left(\frac{1}{E(y+ik)-1} - \frac{1}{E(y-ik)-1} \right) g(k) \\ &= \frac{\pi i}{2m} \sum_{k=1}^{m-1} \left(\frac{E(y-ik)-E(y+ik)}{E(2y)+1-E(y+ik)-E(y-ik)} \right) g(k) \\ &= \frac{\pi i}{2m} \sum_{k=1}^{m-1} \left(\frac{E(-ik)-E(ik)}{E(y)+E(-y)-E(ik)-E(-ik)} \right) g(k) \\ &= \frac{\pi}{2m} \sum_{k=1}^{m-1} \frac{g(k)\sin(2\pi k/m)}{\cosh(2\pi y/m)-\cos(2\pi k/m)}, \end{aligned}$$

which is (4.6). □

Lemma 6. *Let m be a positive integer, $\omega := \exp(2\pi i/m)$, y a positive real number, $g : \mathbf{Z} \to \mathbf{C}$ even and periodic of period m, $M(g)$ as in (1.1) and T_g as in (4.3). Furthermore, let $A_k(y) := \exp(2\pi y/m) - \cos(2\pi k/m)$. Then*

$$\begin{aligned} T_g(0, y) &= \sum_{k=1}^{\infty} \frac{g(k)}{k^2+y^2} \\ &= -\tfrac{1}{2} y^{-2} g(0) + \tfrac{1}{2}\pi y^{-1} m^{-1} \sum_{k=0}^{m-1} g(k) \coth(\pi(y+ik)/m) && (4.7) \\ &= \tfrac{1}{2}\pi y^{-1} M(g) - \tfrac{1}{2} y^{-2} g(0) + \frac{\pi}{ym} \sum_{k=0}^{m-1} \frac{g(k)}{e^{2\pi y/m}\omega^k - 1} && (4.8) \\ &= \tfrac{1}{2}\pi y^{-1} M(g) - \tfrac{1}{2} y^{-2} g(0) - \frac{\pi}{ym} \sum_{k=0}^{m-1} \frac{g(k) A_k(-y)}{A_k(y) + A_k(-y)}. && (4.9) \end{aligned}$$

Proof: Since g is even,

$$\begin{aligned} 2y \sum_{k=1}^{\infty} \frac{g(k)}{k^2+y^2} &= \sum_{k=1}^{\infty} \left(\frac{1}{y+ik} + \frac{1}{y-ik} \right) g(k) \\ &= \sum_{k=1}^{\infty} \left(\frac{g(k)}{y+ik} + \frac{g(-k)}{y-ik} \right) \\ &= P.V. \sum_{0 \neq k \in \mathbf{Z}} \frac{g(k)}{y+ik}. \end{aligned}$$

Formula (4.7) now follows directly from Lemma 4.

If we now write

$$\sum_{k=0}^{m-1} g(k)\coth(\pi(y+ik)/m) = \sum_{k=0}^{m-1} g(k)\left(1+\frac{2}{e^{2\pi(y+ik)/m}-1}\right)$$
$$= mM(g) + \sum_{k=0}^{m-1}\frac{2g(k)}{e^{2\pi y/m}\omega^k-1},$$

we see that formula (4.8) follows from (4.7).

Finally, let $E:\mathbf{C}\to\mathbf{C}$ be defined by $E(z):=\exp(2\pi z/m)$. Then

$$\sum_{k=0}^{m-1}\frac{g(k)}{e^{2\pi y/m}\omega^k-1} = \frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{g(k)}{E(y+ik)-1}+\frac{g(m-k)}{E(y+i(m-k))-1}\right)$$
$$= \frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{1}{E(y+ik)-1}+\frac{1}{E(y-ik)-1}\right)g(k)$$
$$= \frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{E(y-ik)+E(y+ik)-2}{E(2y)+1-E(y+ik)-E(y-ik)}\right)g(k)$$
$$= \frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{E(ik)+E(-ik)-2E(-y)}{E(y)+E(-y)-E(ik)-E(-ik)}\right)g(k)$$
$$= \frac{1}{2}\sum_{k=0}^{m-1}\left(\frac{2\cos(2\pi k/m)-2\exp(-2\pi y/m)}{2\cosh(2\pi y/m)-2\cos(2\pi k/m)}\right)g(k)$$
$$= -\sum_{k=0}^{m-1}\frac{g(k)A_k(-y)}{A_k(y)+A_k(-y)}$$

shows that (4.9) follows from (4.8). □

Lemma 7. *Let m and q be positive integers, x a positive real number, $g:\mathbf{Z}\to\mathbf{C}$ odd and periodic of period m, and $L(s,g)$ as in* (1.2). *Then*

$$\tfrac{1}{2}\pi x L(2q,g) + \pi x\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2\pi n/x}-1} + \sum_{j=0}^{q}(-1)^j x^{-2j+2}\zeta(2j)L(2q-2j+1,g)$$
$$= (-1)^q x^{-2q+2}\pi i m^{-1}\sum_{k=1}^{m-1} g(k)\sum_{n=1}^{\infty}\frac{n^{-2q}}{e^{2\pi nx/m}\omega^k-1}$$
$$= \tfrac{1}{2}(-1)^q x^{-2q+2}\pi m^{-1}\sum_{k=1}^{m-1} g(k)\sin(2\pi k/m)\sum_{n=1}^{\infty}\frac{n^{-2q}}{\cosh(2\pi nx/m)-\cos(2\pi k/m)}.$$

Proof: In Lemma 3, set $s=2q-1$ and let f be the restriction of g to the set $\mathbf{Z}^+$ of positive integers, so that $L(s,g)=F(s)$ and $T_g=T_f$. As in the proof of Lemma 5, since g is odd and has period m, $M(g) = 0$, and thus $\sigma_g = \sigma_f \leq 0$. Therefore, the convergence condition $\Re(s) > \sigma_g + 2q - 2$ of Lemma 3 is satisfied. Substituting the formulas of Lemma 5 for $T_g(-1, nx)$ completes the proof. □

Lemma 8. *Let m and q be positive integers, x a positive real number, $g : \mathbf{Z} \to \mathbf{C}$ even and periodic of period m, $M(g)$ as in* (1.1), *and $L(s, g)$ as in* (1.2). *Then*

$$\begin{aligned}
&\tfrac{1}{2}\pi x L(2q+1, g) + \pi x \sum_{n=1}^{\infty} \frac{n^{-2q-1} g(n)}{e^{2\pi n/x} - 1} + \sum_{j=0}^{q+1} (-1)^j x^{-2j+2} \zeta(2j) L(2q - 2j + 2, g) \\
&= (-1)^q \pi x^{-2q+1} \left\{ \tfrac{1}{2} M(g)\zeta(2q+1) + \frac{1}{m} \sum_{k=0}^{m-1} g(k) \sum_{n=1}^{\infty} \frac{n^{-2q-1}}{e^{2\pi n x/m}\omega^k - 1} \right\} \\
&= \tfrac{1}{2}(-1)^q \pi x^{-2q+1} \Bigg\{ M(g)\zeta(2q+1) \\
&\quad - \frac{1}{m} \sum_{k=0}^{m-1} g(k) \sum_{n=1}^{\infty} \left(\frac{\exp(-2\pi n x/m) - \cos(2\pi k/m)}{\cosh(2\pi n x/m) - \cos(2\pi k/m)} \right) n^{-2q-1} \Bigg\}.
\end{aligned}$$

Proof: In Lemma 3, set $s = 2q$ and let f be the restriction of g to set $\mathbf{Z}^+$ of positive integers. Since $\sigma_g \leq 1$, the convergence condition $\Re(s) > \sigma_g + 2q - 2$ of Lemma 3 is satisfied. Next, substitute the formulas of Lemma 6 for $T_g(0, nx)$. The term

$$-\tfrac{1}{2}(-1)^q x^{-2q} g(0) \zeta(2q+2)$$

arising from the term $-g(0)/2y^2$ in (4.8) and (4.9) can be absorbed into the sum on j, since for even g, $L(0, g) = -g(0)/2$ by Proposition 1. □

Proof of Theorems 1 and 2: Set $x = \pi/\alpha$, $\beta = \pi x = \pi^2/\alpha$ in Lemmas 7 and 8. □

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