PARTITION IDENTITIES FOR THE MULTIPLE ZETA FUNCTION

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Abstract. We define a class of expressions for the multiple zeta function, and show how to determine whether an expression in the class vanishes identically. The class of such identities, which we call partition identities, is shown to coincide with the class of identities that can be derived as a consequence of the stuffle multiplication rule for multiple zeta values.

1. Introduction

For positive integer n and real $s_j \geq 1$ $(j = 1, 2, \ldots, n)$ the multiple zeta function may be defined by

$$\zeta(s_1, s_2, \ldots, s_n) = \sum_{k_1 > k_2 > \cdots > k_n > 0} \prod_{j=1}^{n} k_j^{-s_j}. \tag{1.1}$$

The nested sum (1.1) is over all positive integers $k_1, \ldots, k_n$ satisfying the indicated inequalities, and is finite if and only if $s_1 > 1$ also holds. An elementary property of the multiple zeta function is that it satisfies the so-called stuffle multiplication rule [1]: If $\vec{u} = (u_1, \ldots, u_m)$ and $\vec{v} = (v_1, \ldots, v_n)$, then

$$\zeta(\vec{u})\zeta(\vec{v}) = \sum_{\vec{w} \in \vec{u} * \vec{v}} \zeta(\vec{w}), \tag{1.2}$$

where $\vec{u} * \vec{v}$ is the multi-set of size [2]

$$|\vec{u} * \vec{v}| = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} 2^k$$

defined by the recursion

$$(s, \vec{u}) * (t, \vec{v}) = \{(s, \vec{w}) : \vec{w} \in \vec{u} * (t, \vec{v})\} \cup \{(t, \vec{w}) : \vec{w} \in (s, \vec{u}) * \vec{v}\} \cup \{(s + t, \vec{w}) : \vec{w} \in \vec{u} * \vec{v}\},$$

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with initial conditions $\vec{u} * () = () * \vec{u} = \vec{u}$. Thus, for example,

$$\begin{aligned}(s,u)*(t,v) = &\{(s,u,t,v),(s,u+t,v),(s,t,u,v),(s,t,u+v),(s,t,v,u)\}\\ &\cup\{(t,s,u,v),(t,s,u+v),(t,s,v,u),(t,s+v,u),(t,v,s,u)\}\\ &\cup\{(s+t,u,v),(s+t,u+v),(s+t,v,u)\},\end{aligned}$$

and correspondingly, we have the stuffle identity

$$\begin{aligned}\zeta(s,u)\zeta(t,v) &= \zeta(s,u,t,v)+\zeta(s,u+t,v)+\zeta(s,t,u,v)+\zeta(s,t,u+v)+\zeta(s,t,v,u)\\ &\quad+\zeta(t,s,u,v)+\zeta(t,s,u+v)+\zeta(t,s,v,u)+\zeta(t,s+v,u)+\zeta(t,v,s,u)\\ &\quad+\zeta(s+t,u,v)+\zeta(s+t,u+v)+\zeta(s+t,v,u).\end{aligned}$$

The sum on the right hand side of equation (1.2) accounts for all possible interlacings of the summation indices when the two nested series on the left are multiplied.

In this paper, we consider a certain class of expressions ("legal expressions") for the multiple zeta function, consisting of a finite linear combination of terms. Roughly speaking, a term is a product of multiple zeta functions, each of which is evaluated at a sequence of sums selected from a common argument list $(s_1,\dots,s_n)$ in such a way that each variable s_j appears exactly once in each term. A more precise definition is given in Section 2. Once the legal expressions have been defined, we consider the problem of determining when a legal expression vanishes identically. For reasons which will become clear, we call such identities *partition identities*. It will be seen that the problem of verifying or refuting an alleged partition identity reduces to finite arithmetic over a polynomial ring. Alternatively, one can first rewrite any legal expression as a sum of single multiple zeta functions by applying the stuffle multiplication rule to each term. As we shall see, it is then easy to determine whether or not the original expression vanishes identically.

2. Definitions

Our definition of a partition identity makes use of the concept of a set partition. It is helpful to distinguish between set partitions that are ordered and those that are unordered.

Definition 1 (Unordered Set Partition)**.** Let S be a finite non-empty set. An *unordered* set partition of S is a finite non-empty set $\mathcal{P}$ whose elements are disjoint non-empty subsets of S with union S. That is, there exists a positive integer $m = |\mathcal{P}|$ and non-empty subsets $P_1,\dots,P_m$ of S such that $\mathcal{P} = \{P_1,\dots,P_m\}$, $S = \cup_{k=1}^m P_k$, and $P_j \cap P_k$ is empty if $j \ne k$.

Definition 2 (Ordered Set Partition)**.** Let S be a finite non-empty set. An *ordered* set partition of S is a finite ordered tuple $\vec{P}$ of disjoint non-empty subsets of S such that the union of the components of $\vec{P}$ is equal to S. That is, there exists a positive integer m and non-empty subsets $P_1,\dots,P_m$ of S such that $\vec{P}$ can be identified with the ordered m-tuple $(P_1,\dots,P_m)$, $\cup_{k=1}^m P_k = S$, and $P_j \cap P_k$ is empty if $j \ne k$.

Definition 3 (Legal Term)**.** Let n be a positive integer and let $\vec{s} = (s_1, \ldots, s_n)$ be an ordered tuple of n real variables with $s_j > 1$ for $1 \le j \le n$. Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be an unordered set partition of the first n positive integers $\{1, 2, \ldots, n\}$. For each positive integer k such that $1 \le k \le m$, let $\vec{P}_k = (P_k^{(1)}, P_k^{(2)}, \ldots, P_k^{(\alpha_k)})$ be an ordered set partition of P_k, and let

$$t_k^{(r)} = \sum_{j \in P_k^{(r)}} s_j, \qquad 1 \le r \le \alpha_k = |\vec{P}_k|.$$

A *legal term* for $\vec{s}$ is a product of the form

$$\prod_{k=1}^{m} \zeta\big(t_k^{(1)}, t_k^{(2)}, \ldots, t_k^{(\alpha_k)}\big), \tag{2.1}$$

and every legal term for $\vec{s}$ has the form (2.1) for some unordered set partition $\mathcal{P}$ of $\{1, 2, \ldots, n\}$ and ordered subpartitions $\vec{P}_k$, $1 \le k \le |\mathcal{P}|$.

Example 1. The product $\zeta(s_6, s_2 + s_5, s_1 + s_8 + s_9)\zeta(s_3 + s_4, s_{10})\zeta(s_7)$ is a legal term for the 10-tuple $(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10})$ arising from the partition $\{P_1, P_2, P_3\}$ of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, where $P_1 = \{1, 2, 5, 6, 8, 9\}$ has ordered subpartition $\vec{P}_1 = (\{6\}, \{2, 5\}, \{1, 8, 9\})$, $P_2 = \{3, 4, 10\}$ has ordered subpartition $\vec{P}_2 = (\{3, 4\}, \{10\})$, and $P_3 = \{7\}$ has ordered subpartition $\vec{P}_3 = (\{7\})$.

Definition 4 (Legal Expression)**.** Let n be a positive integer, and let $\vec{s} = (s_1, \ldots, s_n)$ be an ordered tuple of n real variables with $s_j > 1$ for $1 \le j \le n$. A *legal expression* for $\vec{s}$ is a finite $\mathbf{Z}$-linear combination of legal terms for $\vec{s}$. That is, for any positive integer q, integers a_h, and legal terms T_h for $\vec{s}$ $(1 \le h \le q)$, the sum $\sum_{h=1}^{q} a_h T_h$ is a legal expression for $\vec{s}$, and every legal expression for $\vec{s}$ has this form.

Definition 5 (Partition Identity)**.** A *partition identity* is an equation of the form LHS $= 0$ for which there exists a positive integer n and real variables $s_j > 1$ $(j = 1, 2, \ldots, n)$ such that LHS is a legal expression for $(s_1, \ldots, s_n)$, and the equation holds true for all real values of the variables $s_j > 1$.

Example 2. The equation

$$2\zeta(s_1 + s_2 + s_3) - \zeta(s_2)\zeta(s_1 + s_3) - \zeta(s_3)\zeta(s_1 + s_2) + \zeta(s_1 + s_2, s_3) \\ + \zeta(s_2, s_1 + s_3) + \zeta(s_1 + s_3, s_2) + \zeta(s_3, s_1 + s_2) = 0$$

is a partition identity, and is easily verified by expanding the two products $\zeta(s_2)\zeta(s_1 + s_3)$ and $\zeta(s_3)\zeta(s_1 + s_2)$ using the stuffle multiplication rule (1.2) and then collecting multiple zeta functions with identical arguments. A natural question is whether *every* partition identity can be verified in this way. We provide an affirmative answer to this question in Section 4. An alternative method for verifying partition identities is given in Section 3.

3. Rational Functions

Here, we describe a method by which one can determine whether or not a legal expression vanishes identically, or equivalently, whether or not an alleged partition identity is in fact a true identity. It will be seen that the problem reduces to that of checking whether or not an associated rational function identity is true. This latter check can be accomplished in a completely deterministic and mechanical fashion by clearing denominators and expanding the resulting multivariate polynomials. More specifically, we associate rational functions with legal terms in such a way that the alleged partition identity holds if and only if the corresponding rational function identity, in which each legal term is replaced by its associated rational function, holds. The rational function corresponding to (2.1) is the function of n real variables $x_1 > 1, \dots, x_n > 1$ defined by

$$R(x_1, x_2, \dots, x_n) := \prod_{k=1}^{m} \prod_{\beta=1}^{\alpha_k} \biggl(\prod_{\lambda=1}^{\beta} \prod_{j \in P_k^{(\lambda)}} x_j - 1 \biggr)^{-1}. \tag{3.1}$$

Theorem 1. *Let q be a positive integer, and let $E = \sum_{h=1}^{q} a_h T_h$ be a legal expression for $\vec{s} = (s_1, \dots, s_n)$ (i.e. each $a_h \in \mathbf{Z}$ and T_h is a legal term for $\vec{s}$, $1 \le h \le q$). Let $L = \sum_{h=1}^{q} a_h r_h$ be the expression obtained by replacing each legal term T_h by its corresponding rational function according to the rule that associates* (3.1) *with* (2.1)*. Then E vanishes identically if and only if L does.*

Example 3. The rational function identity which Theorem 1 asserts is equivalent to the partition identity of Example 2 is

$$\begin{aligned} &\frac{2}{x_1x_2x_3 - 1} - \frac{1}{x_2 - 1} \cdot \frac{1}{x_1x_3 - 1} - \frac{1}{x_3 - 1} \cdot \frac{1}{x_1x_2 - 1} + \frac{1}{(x_1x_2 - 1)(x_1x_2x_3 - 1)} \\ &\qquad + \frac{1}{(x_2 - 1)(x_1x_2x_3 - 1)} + \frac{1}{(x_1x_3 - 1)(x_1x_2x_3 - 1)} + \frac{1}{(x_3 - 1)(x_1x_2x_3 - 1)} = 0, \end{aligned}$$

which can be readily verified by hand, or with the aid of a suitable computer algebra system.

Proof of Theorem 1. It is immediate from the partition integral [1] representation for the multiple zeta function that every legal term on $(s_1, \dots, s_n)$ is an n-dimensional integral transform of its associated rational function multiplied by the common kernel

$\prod_{j=1}^{n}(\log x_j)^{s_j-1}/\Gamma(s_j)x_j$. Explicitly,

$$\prod_{k=1}^{m}\zeta\Bigl(\sum_{j\in P_k^{(1)}} s_j, \sum_{j\in P_k^{(2)}} s_j, \dots, \sum_{j\in P_k^{(\alpha_k)}} s_j\Bigr)$$
$$= \int_1^\infty \cdots \int_1^\infty \Biggl\{\prod_{k=1}^{m}\prod_{\beta=1}^{\alpha_k}\Bigl(\prod_{\lambda=1}^{\beta}\prod_{j\in P_k^{(\lambda)}} x_j - 1\Bigr)^{-1}\Biggr\}\prod_{j=1}^{n}\frac{(\log x_j)^{s_j-1}}{\Gamma(s_j)x_j}\,dx_j.$$

Linearity of the integral implies that if $L \equiv 0$ then $E \equiv 0$. The real content of Theorem 1 is that converse the also holds. To prove this, we first note that the rational function (3.1) is continuous on the n-fold Cartesian product of open intervals $(1..\infty)^n = \{(x_1,\dots,x_n) \in \mathbf{R}^n : x_j > 1, 1 \le j \le n\}$ and $|R(x_1,\dots,x_n)\prod_{j=1}^{n} x_j|$ is bounded on any n-fold Cartesian product of half-open intervals of the form $[c..\infty)^n = \{(x_1,\dots,x_n) \in \mathbf{R}^n : x_j \ge c, 1 \le j \le n\}$ with $c > 1$. These properties obviously extend to linear combinations of rational functions of the form (3.1), and thus to complete the proof of Theorem 1, it suffices to establish the following result.

Lemma 1. *Let n be a positive integer and let R be a continuous real-valued function of n real variables defined on the n-fold Cartesian product of open intervals $(1..\infty)^n$. Suppose there exists a constant $c > 1$ such that $|R(x_1, x_2, \dots, x_n)\prod_{j=1}^{n} x_j|$ is bounded on the n-fold Cartesian product of half-open intervals $[c..\infty)^n$. Suppose further that there exist non-negative real numbers $s_1^*, s_2^*, \dots, s_n^*$ such that the n-dimensional multiple integral*

$$\int_1^\infty \cdots \int_1^\infty R(x_1, x_2, \dots, x_n)\prod_{j=1}^{n}(\log x_j)^{s_j}\,\frac{dx_j}{x_j}$$

vanishes whenever $s_j > s_j^*$ for $1 \le j \le n$. Then R vanishes identically.

Proof. Fix $s_j > s_j^*$ for $1 \le j \le n$. Let $T : [1..\infty) \to \mathbf{R}$ be given by the convergent $(n-1)$-dimensional multiple integral

$$T(x) := \int_1^\infty \cdots \int_1^\infty R(x_1,\dots,x_{n-1},x)\prod_{j=1}^{n-1}(\log x_j)^{s_j}\,\frac{dx_j}{x_j}.$$

Then $T(x) = O(1/x)$ as $x \to \infty$. It follows that the Laplace Transform

$$F(z) := \int_0^\infty e^{-zu}\,T(e^u)\,du$$

is analytic in the right half-plane $\{z \in \mathbf{C} : \Re(z) > -1\}$, and for all positive integers $m > s_n^*$,

$$\Bigl(-\frac{d}{dz}\Bigr)^m F(z)\Bigr|_{z=0} = \int_0^\infty u^m\,T(e^u)\,du = \int_1^\infty (\log x)^m\,T(x)\,\frac{dx}{x} = 0.$$

By Taylor's theorem, F is a polynomial. Letting $z \to +\infty$ in the definition of F, we see that in fact, F must be the zero polynomial. By the uniqueness theorem for Laplace transforms (see eg. [4]), the set of $x > 1$ for which $T(x) \neq 0$ is of Lebesgue measure zero. Since T is continuous, it follows that $T(x) = 0$ for all $x > 1$. If $n = 1$, then $T = R$ and we're done. Otherwise, fix $x > 1$, and suppose the result holds for $n-1$. Since in the above argument, $s_1 > s_1^*, \ldots, s_{n-1} > s_{n-1}^*$ were arbitrary, $T(x) = 0$ implies $R(x_1, x_2, \ldots, x_{n-1}, x) = 0$ for all $x_1, x_2, \ldots, x_{n-1}$ by the inductive hypothesis. Since this is true for each fixed $x > 1$, the result follows. □

4. Stuffles and Partition Identities

As in [1], we define the class of stuffle identities to be the set of all identities of the form (1.2). In [1], it is shown that every stuffle identity is a consequence of a corresponding rational function identity. In the previous section of the present paper, using a different method of proof, we established the more general result that every partition identity is a consequence of a corresponding rational function identity. Clearly every stuffle identity is a partition identity, but not conversely. Nevertheless, we shall see that every partition identity is a consequence of the stuffle multiplication rule. More specifically, we provide an affirmative answer to the question raised at the end of Example 2 in Section 2.

Notation. We introduce the concatenation operator **Cat**, which will be useful for expressing argument sequences without recourse to ellipses. For example, $\mathbf{Cat}_{k=1}^{m} t_j$ denotes the sequence $t_1, \ldots, t_m$.

As we noted previously, by applying the stuffle multiplication rule (1.2) to legal terms, any legal expression on $(s_1, \ldots, s_n)$ can be rewritten as a finite $\mathbf{Z}$-linear combination of single multiple zeta functions of the form

$$\sum_{h=1}^{q} a_h\, \zeta\Big(\mathop{\mathbf{Cat}}_{k=1}^{\alpha_h} \sum_{j\in P_k} s_j\Big) = \sum_{h=1}^{q} a_h\, \zeta\Big(\sum_{j\in P_1} s_j, \sum_{j\in P_2} s_j, \ldots, \sum_{j\in P_{\alpha_h}} s_j\Big),$$

where the coefficients a_h are integers, q is a positive integer, and $(P_1, \ldots, P_{\alpha_h})$ is an ordered set partition of the first n positive integers $\{1, 2, \ldots, n\}$ for each $h = 1, 2, \ldots, q$. Thus, it suffices to prove the following result.

Theorem 2. *Let F be a finite non-empty set of positive integers, and let $\{s_j : j \in F\}$ be a set of real variables, each exceeding 1. Suppose that for all $s_j > 1$,*

$$\sum_{\vec{P}\vDash F} c_{\vec{P}}\, \zeta\Big(\mathop{\mathbf{Cat}}_{k=1}^{|\vec{P}|} \sum_{j\in P_k} s_j\Big) = 0, \tag{4.1}$$

where the sum is over all ordered set partitions $\vec{P}$ of F, P_k denotes the k^{th} component of $\vec{P}$, and the coefficients $c_{\vec{P}}$ are real numbers depending only on $\vec{P}$. Then each $c_{\vec{P}} = 0$.

Proof. We argue by induction on the cardinality of F, the case $|F| = 1$ being trivial. To clarify the argument, we present the cases $|F| = 2$ and $|F| = 3$ before proceeding to the inductive step.

When $|F| = 2$, the identity (4.1) takes the form

$$c_1\zeta(s,t) + c_2\zeta(t,s) + c_3\zeta(s+t) = 0, \qquad s > 1,\ t > 1.$$

By Theorem 1, this is equivalent to the rational function identity

$$\frac{c_1}{(x-1)(xy-1)} + \frac{c_2}{(y-1)(xy-1)} + \frac{c_3}{xy-1} = 0, \qquad x > 1,\ y > 1.$$

Letting $x \to 1+$ shows that we must have

$$\frac{c_1}{y-1} = 0 \quad \Longrightarrow \quad c_1\zeta(t) = 0 \quad \Longrightarrow \quad c_1 = 0.$$

Similarly, letting $y \to 1+$ shows that $c_2 = 0$. Since the remaining term must vanish, we must have $c_3 = 0$ as well.

When $|F| = 3$, the identity (4.1) takes the form

$$\begin{aligned} 0 &= c_1\zeta(s,t,u) + c_2\zeta(s,u,t) + c_3\zeta(t,s,u) + c_4\zeta(t,u,s) + c_5\zeta(u,s,t) + c_6\zeta(u,t,s) \\ &\quad + c_7\zeta(s,t+u) + c_8\zeta(t+u,s) + c_9\zeta(t,s+u) + c_{10}\zeta(s+u,t) + c_{11}\zeta(u,s+t) \\ &\quad + c_{12}\zeta(s+t,u) + c_{13}\zeta(s+t+u), \qquad s > 1,\ t > 1,\ u > 1, \end{aligned}$$

which, by Theorem 1, is equivalent to the rational function identity

$$\begin{aligned} 0 &= \frac{c_1}{(x-1)(xy-1)(xyz-1)} + \frac{c_2}{(x-1)(xz-1)(xyz-1)} + \frac{c_3}{(y-1)(xy-1)(xyz-1)} \\ &+ \frac{c_4}{(y-1)(yz-1)(xyz-1)} + \frac{c_5}{(z-1)(xz-1)(xyz-1)} + \frac{c_6}{(z-1)(yz-1)(xyz-1)} \\ &+ \frac{c_7}{(x-1)(xyz-1)} + \frac{c_8}{(yz-1)(xyz-1)} + \frac{c_9}{(y-1)(xyz-1)} + \frac{c_{10}}{(xz-1)(xyz-1)} \\ &+ \frac{c_{11}}{(z-1)(xyz-1)} + \frac{c_{12}}{(xy-1)(xyz-1)} + \frac{c_{13}}{xyz-1}, \qquad x > 1,\ y > 1,\ z > 1. \end{aligned} \tag{4.2}$$

If we let $x \to 1+$ in (4.2), then for the singularities to cancel, we must have

$$0 = \frac{c_1}{(y-1)(yz-1)} + \frac{c_2}{(z-1)(yz-1)} + \frac{c_7}{yz-1}, \qquad y > 1,\ z > 1,$$

which, in light of Theorem 1, implies the identity

$$0 = c_1\zeta(t,u) + c_2\zeta(u,t) + c_7\zeta(t+u), \qquad t > 1,\ u > 1.$$

Having proved the $|F| = 2$ case of our result, we see that this implies $c_1 = c_2 = c_7 = 0$. Similarly, letting $y \to 1+$ in (4.2) gives $c_3 = c_4 = c_9 = 0$, and letting $z \to 1+$ in (4.2) gives $c_5 = c_6 = c_{11} = 0$. At this point, only 4 terms in (4.2) remain. Letting $yz \to 1+$ now shows that $c_8 = 0$, and the remaining coefficients can be shown to vanish similarly.

For the inductive step, let $|F|>1$ and suppose Theorem 2 is true for all non-empty sets of positive integers of cardinality less than $|F|$. Suppose also that (4.1) holds. By Theorem 1, it follows that the rational function identity

$$\sum_{\vec{P}\vDash F} c_{\vec{P}} \prod_{m=1}^{|\vec{P}|} \Big(\prod_{k=1}^{m}\prod_{j\in P_k} x_j - 1\Big)^{-1} = 0 \tag{4.3}$$

holds for all $x_j>1$, $j\in F$. Fix $f\in F$ and let $x_f\to 1+$ in (4.3). For the singularities to cancel, we must have

$$\sum_{\substack{\vec{P}\vDash F\\ P_1=\{f\}}} c_{\vec{P}} \prod_{m=2}^{|\vec{P}|} \Big(\prod_{k=2}^{m}\prod_{j\in P_k} x_j - 1\Big)^{-1} = 0,$$

which, by Theorem 1, implies that

$$\sum_{\substack{\vec{P}\vDash F\\ P_1=\{f\}}} c_{\vec{P}}\, \zeta\Big(\operatorname*{\mathbf{Cat}}_{m=2}^{|\vec{P}|} \sum_{j\in P_m} s_j\Big) = \sum_{\vec{P}\vDash F\setminus\{f\}} c_{\vec{P}}\, \zeta\Big(\operatorname*{\mathbf{Cat}}_{m=1}^{|\vec{P}|} \sum_{j\in P_m} s_j\Big) = 0.$$

By the inductive hypothesis, $c_{\vec{P}}=0$ for every ordered set partition $\vec{P}$ of F whose first component P_1 is the singleton $\{f\}$. Since $f\in F$ was arbitrary, it follows that $c_{\vec{P}}=0$ for every ordered set partition $\vec{P}$ of F whose first component P_1 consists of a single element.

Proceeding inductively, suppose we've shown that $c_{\vec{P}}=0$ for every ordered set partition $\vec{P}$ of F with $|P_1|=r-1<|F|$. Let G be a subset of F of cardinality r. If in (4.3) we now let $x_g\to 1+$ for each $g\in G$, then as the singularities in the remaining terms (4.3) must cancel, we must have

$$\sum_{\substack{\vec{P}\vDash F\\ P_1=G}} c_{\vec{P}} \prod_{m=2}^{|\vec{P}|} \Big(\prod_{k=2}^{m}\prod_{j\in P_k} x_j - 1\Big)^{-1} = 0.$$

Theorem 1 then implies that

$$\sum_{\substack{\vec{P}\vDash F\\ P_1=G}} c_{\vec{P}}\, \zeta\Big(\operatorname*{\mathbf{Cat}}_{m=2}^{|\vec{P}|} \sum_{j\in P_m} s_j\Big) = \sum_{\vec{P}\vDash F\setminus G} c_{\vec{P}}\, \zeta\Big(\operatorname*{\mathbf{Cat}}_{m=1}^{|\vec{P}|} \sum_{j\in P_m} s_j\Big) = 0.$$

By the inductive hypothesis, $c_{\vec{P}}=0$ for every ordered set partition $\vec{P}$ of F with first component equal to G. Since G was an arbitrary subset of F of cardinality r, it follows that $c_{\vec{P}}=0$ for every ordered set partition $\vec{P}$ of F whose first component has cardinality r. By induction on r, it follows that $c_{\vec{P}}=0$ for every ordered set partition $\vec{P}$ of F, as claimed. □

Since every partition identity is thus a consequence of the stuffle multiplication rule, it follows that any multi-variate function that obeys a stuffle multiplication rule will satisfy every partition identity satisfied by the multiple zeta function. For example, suppose we fix a positive integer N and a set F of functions $f : \mathbf{Z}^+ \to \mathbf{C}$ closed under point-wise addition. For positive integer n and $f_1, \dots, f_n \in F$, define

$$z_N(f_1, f_2, \dots, f_n) := \sum_{N > k_1 > k_2 > \cdots > k_n > 0} \prod_{j=1}^{n} \exp(f_j(k_j)),$$

where the sum is over all positive integers $k_1, \dots, k_n$ satisfying the indicated inequalities. Then for all $g, h \in F$, we have $z_N(g)z_N(h) = z_N(g, h) + z_N(h, g) + z_N(g + h)$, and more generally, if $\vec{g} = (g_1, \dots, g_m)$ and $\vec{h} = (h_1, \dots, h_n)$ are vectors of functions in F, then z_N obeys the stuffle multiplication rule

$$z_N(\vec{g})z_N(\vec{h}) = \sum_{\vec{f} \in \vec{g} * \vec{h}} z_N(\vec{f}).$$

We assert that z_N satisfies every partition identity satisfied by ζ. For example, let n be a positive integer, and let $\mathfrak{S}_n$ denote the group of $n!$ permutations of the first n positive integers $\{1, 2, \dots, n\}$. Let $s_1, \dots, s_n$ be real variables, each exceeding 1. Using a counting argument, Hoffman [3] proved the partition identity

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta\big(\operatorname*{\mathbf{Cat}}_{k=1}^{n} s_{\sigma(k)} \big) = \sum_{\mathcal{P} \vdash \{1, \dots, n\}} (-1)^{n - |\mathcal{P}|} \prod_{P \in \mathcal{P}} (|P| - 1)! \, \zeta\big(\sum_{j \in P} s_j \big), \tag{4.4}$$

in which the sum on the right extends over all unordered set partitions $\mathcal{P}$ of the first n positive integers $\{1, 2, \dots, n\}$, and of course $|\mathcal{P}|$ denotes the number of parts in the partition $\mathcal{P}$. In light of Theorem 2, it follows that Hoffman's identity (4.4) depends on only the stuffle multiplication property (1.2) of the multiple zeta function; whence any function satisfying a stuffle multiplication rule will also satisfy (4.4). In particular, with z_N defined as above,

$$\sum_{\sigma \in \mathfrak{S}_n} z_N\big(\operatorname*{\mathbf{Cat}}_{k=1}^{n} f_{\sigma(k)} \big) = \sum_{\mathcal{P} \vdash \{1, \dots, n\}} (-1)^{n - |\mathcal{P}|} \prod_{P \in \mathcal{P}} (|P| - 1)! \, z_N\big(\sum_{j \in P} f_j \big).$$

Finally, we note that by Theorem 1, the rational function identity

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{k=1}^{n} \Big(\prod_{j=1}^{k} x_{\sigma(j)} - 1 \Big)^{-1} = \sum_{\mathcal{P} \vdash \{1, \dots, n\}} (-1)^{n - |\mathcal{P}|} \prod_{P \in \mathcal{P}} (|P| - 1)! \Big(\prod_{j \in P} x_j - 1 \Big)^{-1}$$

is equivalent to (4.4).

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