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ON MORDELL-TORNHEIM SUMS AND MULTIPLE ZETA VALUES

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To Paulo Ribenboim, in honor of his 80th birthday.

RÉSUMÉ. Nous prouvons que toute somme de Mordell-Tornheim avec des arguments entiers positifs peut s'écrire comme une combinaison linéaire rationnelle de valeurs prises par des fonctions multi-zêta ayant le même poids et la même profondeur. Selon un résultat de Tsumura, il s'ensuit que toute somme de Mordell-Tornheim ayant un poids et une profondeur de parité différente peut s'exprimer comme une combinaison linéaire rationnelle de produits de valeurs prises par des fonctions multi-zêta de profondeur plus petite.

ABSTRACT. We prove that any Mordell-Tornheim sum with positive integer arguments can be expressed as a rational linear combination of multiple zeta values of the same weight and depth. By a result of Tsumura, it follows that any Mordell-Tornheim sum with weight and depth of opposite parity can be expressed as a rational linear combination of products of multiple zeta values of lower depth.

1. Introduction

Let r and w be positive integers, and let $s_1,\dots,s_r$ and s be complex numbers satisfying $s_1+\cdots+s_r+s=w$. A *Mordell-Tornheim sum of depth r and weight w* is a multiple series of the form

$$T(s_1,\dots,s_r;s) := \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^s}. \tag{1}$$

Denote the real part of s by σ, and the real part of s_j by σ_j for $1\le j\le r$. Since (1) remains unchanged if the arguments $s_1,\dots,s_r$ are permuted, we may as well suppose that they are arranged in order of increasing real part. Then $\sigma_1\le\sigma_2\le\cdots\le\sigma_r$, and by Theorem 2.2 below, the series (1) is absolutely convergent if

$$\sigma+\sum_{j=1}^{k}\sigma_j>k$$

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for each $k = 1, 2, \dots, r$. We call (1) a Mordell-Tornheim zeta value in the case when the arguments are all integers. These were first investigated by Tornheim [19] in the case $r = 2$, and later by Mordell [18] and Hoffman [14] with $s_1 = \cdots = s_r = 1$.

Of greater theoretical importance are the so-called multiple zeta series of depth r and weight $w = s_1 + \cdots + s_r$ of the form

$$\zeta(s_1, \dots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \prod_{j=1}^{r} n_j^{-s_j}, \tag{2}$$

in which the sum is over all positive integers $n_1, \dots, n_r$ such that $n_j > n_{j+1}$ for $1 \le j \le r - 1$. By Theorem 2.1 below, the series (2) is absolutely convergent if the partial sums of the real parts of the arguments satisfy

$$\sum_{j=1}^{k} \Re(s_j) > k$$

for each $k = 1, 2, \dots, r$. If $s_1, \dots, s_r$ are all integers, then (2) is called a *multiple zeta value of depth* r *and weight* $s_1 + \cdots + s_r$. Multiple zeta values have been studied extensively; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] for example.

In this paper, we show how to express an arbitrary Mordell-Tornheim zeta value in terms of multiple zeta values of the same weight and depth. More precisely, we shall prove the following result.

Theorem 1.1. *Every Mordell-Tornheim zeta value of depth* r *and weight* w *can be expressed as a rational linear combination of multiple zeta values of depth* r *and weight* w.

Theorem 1.1 shows that the study of Mordell-Tornheim zeta values reduces to the study of multiple zeta values. For example, the following parity result is an immediate consequence of Theorem 1.1 and the corresponding parity result for multiple zeta values due to Tsumura [21] and for which an independent proof is given in [16].

Corollary 1.2. *Every Mordell-Tornheim zeta value of depth at least 2 and with weight and depth of opposite parity can be expressed as a rational linear combination of products of multiple zeta values of lower depth.*

We note that the case $r = 2$ of Theorem 1.2 was proved by Tornheim [19]. Explicit formulas for Tornheim's reduction were given in [15]; see also [23].

2. Convergence criteria

Theorem 2.1. *Let* r *be a positive integer, and let* $s_1, \dots, s_r$ *be complex numbers with respective real parts* $\sigma_1, \dots, \sigma_r$. *The multiple zeta series* (2) *is absolutely convergent if for each positive integer* k *such that* $1 \le k \le r$, *the inequality*

$$\sum_{j=1}^{k} \sigma_j > k$$

holds.

Proof. The case $r=1$ is a familiar consequence of the integral test from calculus. Let d be a positive integer, and let $s_1, s_2, \dots, s_{d+1}$ be complex numbers with respective real parts $\sigma_1, \sigma_2, \dots, \sigma_{d+1}$. First, suppose that $\sigma_{d+1} < 1$. The Euler-Maclaurin sum formula implies that

$$\sum_{n_1>\cdots>n_{d+1}>0} \left| \prod_{j=1}^{d+1} n_j^{-s_j} \right| \ll \sum_{n_1>\cdots>n_d>0} n_d^{1-\sigma_{d+1}} \prod_{j=1}^{d} n_j^{-\sigma_j}. \tag{3}$$

By induction, the series on the right-hand side of (3) converges if for each positive integer k such that $1 \le k \le d-1$,

$$\sum_{j=1}^{k} \sigma_j > k \quad \text{and} \quad \left[(\sigma_d + \sigma_{d+1} - 1) + \sum_{j=1}^{d-1} \sigma_j > d \iff \sum_{j=1}^{d+1} \sigma_j > d+1 \right].$$

Therefore, the series obtained by removing the absolute value bars from the series on the left-hand side of (3) is absolutely convergent *a fortiori* if for each positive integer k such that $1 \le k \le d+1$,

$$\sum_{j=1}^{k} \sigma_j > k.$$

Now suppose that $\sigma_{d+1} \ge 1$ and that

$$\sum_{j=1}^{k} \sigma_j > k$$

for every positive integer k such that $1 \le k \le d+1$. Let $\varepsilon > 0$ be defined by the equation

$$\sum_{j=1}^{d} \sigma_j = d + 2\varepsilon.$$

The Euler-Maclaurin sum formula implies that

$$\begin{aligned} \sum_{n_1>\cdots>n_{d+1}>0} \left| \prod_{j=1}^{d+1} n_j^{-s_j} \right| &\ll \sum_{n_1>\cdots>n_d>0} (\log n_d) \prod_{j=1}^{d} n_j^{-\sigma_j} \\ &\ll \sum_{n_1>\cdots>n_d>0} n_d^{-(\sigma_d-\varepsilon)} \prod_{j=1}^{d-1} n_j^{-\sigma_j}. \end{aligned} \tag{4}$$

By induction, the series on the right-hand side of (4) converges because for each positive integer k such that $1 \le k \le d-1$,

$$\sum_{j=1}^{k} \sigma_j > k \quad \text{and} \quad (\sigma_d - \varepsilon) + \sum_{j=1}^{d-1} \sigma_j = d + \varepsilon > d.$$

Therefore, the series obtained by removing the absolute value bars from the series on the left-hand side of (4) is absolutely convergent. □

Remark 2.1. The condition for absolute convergence of (2) is incorrectly stated in [22] as

$$\sigma_1 > 1 \quad \text{and} \quad \sum_{j=1}^{r} \sigma_j > r.$$

For a counterexample, these inequalities are satisfied if $s_1 = s_3 = 2$ and $s_2 = 0$, but

$$\sum_{n_1 > n_2 > n_3 > 0} n_1^{-2} n_3^{-2} = \sum_{n_1=1}^{\infty} \frac{1}{n_1^2} \sum_{n_2=1}^{n_1-1} \sum_{n_3=1}^{n_2-1} \frac{1}{n_3^2} \geq \sum_{n=3}^{\infty} \frac{1}{n^2} \sum_{k=2}^{n-1} 1 = \sum_{n=3}^{\infty} \frac{n-2}{n^2} = \infty.$$

Sufficient conditions for absolute convergence of a more general class of multiple Dirichlet series are given in [17], but the proof takes 4 pages.

Theorem 2.2. *Let r be a positive integer, and $s_1, \ldots, s_r$ complex numbers arranged so that their respective real parts $\sigma_1, \ldots, \sigma_r$ satisfy $\sigma_j \leq \sigma_{j+1}$ for $1 \leq j \leq r-1$. Let s be a complex number with real part σ. If for each positive integer k such that $1 \leq k \leq r$, the inequality*

$$\sigma + \sum_{j=1}^{k} \sigma_j > k$$

holds, then the Mordell-Tornheim series (1) *is absolutely convergent.*

Proof. The summation indices $m_1, \ldots, m_r$ in (1) obviously satisfy

$$\max\{m_j : 1 \leq j \leq r\} \leq \sum_{j=1}^{r} m_j \leq r \max\{m_j : 1 \leq j \leq r\}.$$

Therefore, using the symbol $\asymp$ as a short-hand for "has the same order of magnitude as", we have

$$\text{(5)} \quad \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \left| \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s} \right| \asymp \sum_{\pi \in \mathfrak{S}_r} \sum_{m_{\pi(1)} > \cdots > m_{\pi(r)} > 0} m_{\pi(1)}^{-\sigma} \prod_{j=1}^{r} m_{\pi(j)}^{-\sigma_{\pi(j)}},$$

where the outer sum on the right is over all permutations π of $\{1, 2, \ldots, r\}$ and we have ignored all cases where there exists an equality between two or more indices m_j since these series converge under less stringent conditions. By Theorem 2.1, the series on the right-hand side of (5) is absolutely convergent if for each permutation π and each positive integer k such that $1 \leq k \leq r$, we have

$$\sigma + \sum_{j=1}^{k} \sigma_{\pi(j)} > k.$$

Since $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_r$, this will clearly be the case if for each positive integer k such that $1 \leq k \leq r$, we have

$$\sigma + \sum_{j=1}^{k} \sigma_j > k. \qquad \square$$

3. Proof of Theorem 1.1

Key to our proof of Theorem 1.1 is the following partial fraction decomposition.

Lemma 3.1. *Let r and $s_1, s_2, \ldots, s_r$ be positive integers, and let $x_1, x_2, \ldots, x_r$ be non-zero real numbers such that $x := x_1 + x_2 + \cdots + x_r \neq 0$. Then*

$$\prod_{j=1}^{r} x_j^{-s_j} = \sum_{j=1}^{r} \left(\prod_{\substack{k=1 \\ k\neq j}}^{r} \sum_{a_k=0}^{s_k-1} \right) M_j \, x^{-s_j-A_j} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{a_k-s_k},$$

where

$$M_j := \frac{(s_j + A_j - 1)!}{(s_j-1)!} \prod_{\substack{k=1 \\ k\neq j}}^{r} \frac{1}{a_k!} \qquad \textit{and} \qquad A_j := \sum_{\substack{k=1 \\ k\neq j}}^{r} a_k.$$

Proof. Applying the partial differential operator

$$\prod_{n=1}^{r} \frac{1}{(s_n-1)!} \left(-\frac{\partial}{\partial x_n} \right)^{s_n-1}$$

to both sides of the trivial identity

$$\prod_{j=1}^{r} x_j^{-1} = \sum_{j=1}^{r} x^{-1} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{-1}, \quad \text{with} \quad x := \sum_{j=1}^{r} x_j$$

yields

$$\begin{aligned} \prod_{j=1}^{r} x_j^{-s_j} &= \sum_{j=1}^{r} \left\{ \prod_{\substack{n=1 \\ n\neq j}}^{r} \frac{1}{(s_n-1)!} \left(-\frac{\partial}{\partial x_n} \right)^{s_n-1} \right\} x^{-s_j} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{-1} \\ &= \sum_{j=1}^{r} \left(\prod_{\substack{k=1 \\ k\neq j}}^{r} \sum_{a_k=0}^{s_k-1} \right) \left(\frac{(s_j + A_j - 1)!}{(s_j-1)!} \prod_{\substack{k=1 \\ k\neq j}}^{r} \frac{1}{a_k!} \right) x^{-s_j-A_j} \prod_{\substack{k=1 \\ k\neq j}}^{r} x_k^{a_k-s_k}, \end{aligned}$$

as claimed. □

Proof of Theorem 1.1. For $1 \le l \le r-1$, let

$$T_l(s_1, \ldots, s_r) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \left(\prod_{k=1}^{l} m_k^{-s_k} \right) \left(\prod_{k=l+1}^{r} n_k^{-s_k} \right), \quad \text{with} \quad n_k := \sum_{j=1}^{k} m_j.$$

In Lemma 3.1, let $x_j = m_j$, multiply both sides by n_r^{-s} and sum over all positive integers m_j for $1 \le j \le r$. We find that

$$(6)\qquad \begin{aligned} T(s_1,\ldots,s_r;s) &= \sum_{j=1}^{r}\left(\prod_{\substack{k=1\\k\neq j}}^{r}\sum_{a_k=0}^{s_k-1}\right) M_j \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} n_r^{-s-s_j-A_j}\prod_{\substack{k=1\\k\neq j}}^{r} m_k^{a_k-s_k} \\ &= \sum_{j=1}^{r}\left(\prod_{\substack{k=1\\k\neq j}}^{r}\sum_{a_k=0}^{s_k-1}\right) M_j T_{r-1}\left(\operatorname*{\mathbf{Cat}}_{\substack{k=1\\k\neq j}}^{r}\{s_k-a_k\}, s+s_j+A_j\right), \end{aligned}$$

where

$$\operatorname*{\mathbf{Cat}}_{\substack{k=1\\k\neq j}}^{r}\{t_k\}$$

abbreviates the concatenated argument sequence $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{r-1}, t_r$. Note that the weight of T in (6) is equal to the sum of the arguments in T_{r-1} on the right-hand side. Now apply Lemma 3.1 with $r = l$, $x_j = m_j$, multiply both sides by

$$\prod_{k=l+1}^{r} n_k^{-s_k}$$

and sum over all positive integers m_j for $1 \le j \le r$. We find that

$$(7)\qquad \begin{aligned} T_l(s_1,\ldots,s_r) &= \sum_{j=1}^{l}\left(\prod_{\substack{k=1\\k\neq j}}^{l}\sum_{a_k=0}^{s_k-1}\right) M_j \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\left(\prod_{\substack{k=1\\k\neq j}}^{l} m_k^{a_k-s_k}\right) n_l^{-s_j-A_j}\prod_{k=l+1}^{r} n_k^{-s_k} \\ &= \sum_{j=1}^{l}\left(\prod_{\substack{k=1\\k\neq j}}^{l}\sum_{a_k=0}^{s_k-1}\right) M_j T_{l-1}\left(\operatorname*{\mathbf{Cat}}_{\substack{k=1\\k\neq j}}^{l}\{s_k-a_k\}, s_j+A_j, \operatorname*{\mathbf{Cat}}_{k=l+1}^{r} s_k\right). \end{aligned}$$

Since

$$\sum_{\substack{k=1\\k\neq j}}^{l}(s_k-a_k) + s_j + A_j + \sum_{k=l+1}^{r} s_k = \sum_{k=1}^{r} s_k,$$

the weight is preserved in (7). Since

$$\begin{aligned} T_1(s_1,\ldots,s_r) &= \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}\prod_{k=2}^{r} n_k^{-s_k} \\ &= \sum_{n_r>\cdots>n_1>0}\prod_{k=1}^{r} n_k^{-s_k} = \zeta(s_r,\ldots,s_1), \end{aligned}$$

by induction the proof is complete. □

Before concluding, we note the following easy consequences of the results proved in this section.

Corollary 3.2. *Let $r-1$ and s_j-1 be positive integers for $1\le j\le r$. Let M_j and A_j be as in Lemma 3.1 and let T_{r-1} be as in the proof of Theorem 1.1. Then*

$$\prod_{j=1}^{r}\zeta(s_j)=\sum_{j=1}^{r}\left(\prod_{\substack{k=1\\k\neq j}}^{r}\sum_{a_k=0}^{s_k-1}\right)M_jT_{r-1}\left(\mathop{\mathrm{Cat}}_{\substack{k=1\\k\neq j}}^{r}\{s_k-a_k\},s_j+A_j\right).$$

Proof. Sum both sides of Lemma 3.1 over all positive integers $x_1,\dots,x_r$. □

Note that when $r=2$, Corollary 3.2 reduces to Euler's decomposition [12], namely

$$\zeta(s)\zeta(t)=\sum_{a=0}^{s-1}\binom{a+t-1}{t-1}\zeta(t+a,s-a)+\sum_{a=0}^{t-1}\binom{a+s-1}{s-1}\zeta(s+a,t-a).$$

Corollary 3.3 (Corollary 4.2 in [18]**).** *For any positive integers r and s,*

$$T(\underbrace{1,1,\dots,1}_{r};s)=r!\,\zeta(s+1,\underbrace{1,\dots,1}_{r-1}).$$

Now combining equations (30) and (31) in [3], we have

$$T(\underbrace{1,1,\dots,1}_{r};1)=r!\,\zeta(2,\underbrace{1,\dots,1}_{r-1})=r!\,\zeta(r+1)$$

and

$$T(\underbrace{1,1,\dots,1}_{r};2)=r!\,\zeta(3,\underbrace{1,\dots,1}_{r-1})=r!\,\left\{\frac{r+1}{2}\zeta(r+2)-\frac{1}{2}\sum_{k=1}^{r-1}\zeta(k+1)\zeta(r+1-k)\right\}.$$

4. Parity results

In the introductory section, we alluded to the following parity result for multiple zeta values due to Tsumura [21] and for which an independent proof is given in [16].

Theorem 4.1. *Every multiple zeta value of depth at least two and with weight and depth of opposite parity can be expressed as a rational linear combination of products of multiple zeta values of lower depth.*

Clearly, Corollary 1.2 is an immediate consequence of Theorem 4.1 and our Theorem 1.1. Alternatively, we can prove Corollary 1.2 by employing instead a recent parity result of Tsumura [20] for Mordell-Tornheim zeta values.

Theorem 4.2. *Every Mordell-Tornheim zeta value of depth at least two and with weight and depth of opposite parity can be expressed as a rational linear combination of products of Mordell-Tornheim zeta values of lower depth.*

Corollary 1.2 is clearly also an immediate consequence of Tsumura's Theorem 4.2 and our Theorem 1.1.

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