The Harmonic Series and the nth Term Test for Divergence

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The harmonic series $\sum_{n=1}^{\infty} 1/n$ is a popular example of a divergent series whose terms tend to zero. Another example is $\sum_{n=1}^{\infty} \log(1+1/n)$, whose partial sums are unbounded because they telescope to $\log n$:

$$\sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n-1} \log\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n-1} \big(\log(k+1) - \log k\big) = \log n.$$

This telescoping series can be used to show that the harmonic series diverges. Start with the classical inequality $x \geq \log(1+x)$, which is valid for all $x > -1$ (compare the derivatives of both sides and integrate from 0 to x). Now put $x = 1/k$ with $k = 1, 2, 3, \ldots, n-1$ and add the resulting inequalities, obtaining

$$\sum_{k=1}^{n-1} \frac{1}{k} \geq \sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) = \log n.$$