# A $q$-Analog of Euler's Reduction Formula for the Double Zeta Function

David M. Bradley and Xia Zhou

**Abstract** The double zeta function is a function of two arguments defined by a double Dirichlet series, and was first studied by Euler in response to a letter from Goldbach in 1742. By calculating many examples, Euler inferred a closed form evaluation of the double zeta function in terms of values of the Riemann zeta function, in the case when the two arguments are positive integers with opposite parity. Here, we establish a $q$-analog of Euler's evaluation. That is, we state and prove a 1-parameter generalization that reduces to Euler's evaluation in the limit as the parameter $q$ tends to 1.

## 1 Introduction

The double zeta function is defined by

$$\zeta(s,t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \qquad \Re(s) > 1, \qquad \Re(s+t) > 2. \tag{1}$$

The sums (1), and more generally those of the form

$$\zeta(s_1, s_2, \ldots, s_m) := \sum_{k_1 > k_2 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{1}{k_j^{s_j}}, \quad \sum_{j=1}^{n} \Re(s_j) > n, \quad n = 1, 2, \ldots, m, \tag{2}$$

David M. Bradley
Department of Mathematics & Statistics, University of Maine, 5752 Neville Hall, Orono, ME 04469-5752 U.S.A. e-mail: dbradley@member.ams.org

Xia Zhou
Department of Mathematics, Zhejiang University, Hangzhou, 310027 People's Republic of China e-mail: xiazhou0821@hotmail.com

have attracted increasing attention in recent years; see eg. [2, 3, 4, 5, 7, 8, 9, 10, 12, 15, 20]. The survey articles [6, 16, 22, 25] provide an extensive list of references. In (2) the sum is over all positive integers $k_1,\dots,k_m$ satisfying the indicated inequalities. Note that with positive integer arguments, $s_1>1$ is necessary and sufficient for convergence. As is now customary, we refer to the parameter $m$ in (2) as the depth. Of course (2) reduces to the familiar Riemann zeta function when the depth $m=1$.

The problem of evaluating sums of the form (1) with integers $s>1$ and $t>0$ seems to have been first proposed in a letter from Goldbach to Euler [18] in 1742. (See also [17, 19] and [1, p. 253].) Calculating several examples led Euler to infer a closed form evaluation of the double zeta function in terms of values of the Riemann zeta function, in the case when the two arguments have opposite parity. Euler's evaluation can be expressed as follows. Let $s-1$ and $t-1$ be positive integers with opposite parity (i.e. $s+t$ is odd) and let $2h=\max(s,t)$. Then

$$\begin{aligned}\zeta(s,t) &= (-1)^{s+1}\sum_{k=1}^{h}\left[\binom{s+t-2k-1}{t-1}+\binom{s+t-2k-1}{s-1}\right]\zeta(2k)\zeta(s+t-2k)\\ &\quad+\frac{1}{2}\big((1+(-1)^s\big)\zeta(s)\zeta(t)+\frac{1}{2}\left[(-1)^s\binom{s+t}{s}-1\right]\zeta(s+t). \qquad (3)\end{aligned}$$

If we interpret $\zeta(1)=0$, then Euler's formula (3) gives true results also when $t=1$ and $s$ is even, but this case is subsumed by another formula of Euler, namely

$$2\zeta(s,1)=s\zeta(s+1)-\sum_{k=2}^{s-1}\zeta(k)\zeta(s+1-k), \qquad (4)$$

which is valid for all integers $s>1$.

The evaluations (3) and (4) are both examples of *reduction* formulas, since they both give a closed-form evaluation of a sum of depth 2 in terms of sums of depth 1. More generally (see eg. [7, 8]) a reduction formula expresses an instance of (2) in terms of lower depth sums.

With the general goal of gaining a more complete understanding of the myriad relations satisfied by the multiple zeta functions (2) in mind, a $q$-analog of (2) was introduced in [11] and independently in [21] and [23] as

$$\zeta[s_1,s_2,\dots,s_m]:=\sum_{k_1>k_2>\cdots>k_m>0}\prod_{j=1}^{m}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}, \qquad (5)$$

where $0<q<1$ and for any integer $k$,

$$[k]_q:=\frac{1-q^k}{1-q}.$$

Observe that we now have

$$\zeta(s_1,\dots,s_m)=\lim_{q\to 1}\zeta[s_1,\dots,s_m],$$

so that (5) represents a generalization of (2). The papers [11, 12, 14, 13] consider values of the multiple $q$-zeta functions (5) and establish several infinite classes of relations satisfied by them. In particular, the following $q$-analog of (4) was established.

**Theorem 1 (Corollary 8 of [11]).** *Let $s-1$ be a positive integer. Then*

$$2\zeta[s,1] = s\,\zeta[s+1] + (1-q)(s-2)\zeta[s] - \sum_{k=2}^{s-1} \zeta[k]\zeta[s+1-k].$$

Here, we continue this general program of study by establishing a $q$-analog of Euler's reduction formula (3). Throughout the remainder of this paper, $s$ and $t$ denote positive integers with additional restrictions noted where needed, and $q$ is real with $0 < q < 1$.

## 2 $q$-analog of Euler's reduction formula

Throughout this section, we assume $s > 1$. We've seen that $\zeta[s,t]$ as given by (5) is a $q$-analog of $\zeta(s,t)$ in (1). Here, we introduce additional $q$-analogs of $\zeta(s,t)$ by defining

$$\zeta_1[s,t] = \zeta_1[s,t;q] := (-1)^t \sum_{u>v>0} \frac{q^{(s-1)u+(t-1)(-v)}}{[u]_q^s[-v]_q^t} = \sum_{u>v>0} \frac{q^{(s-1)u+v}}{[u]_q^s[v]_q^t}$$

and

$$\begin{aligned}\zeta_2[s,t] = \zeta_2[s,t;q] &:= (-1)^s \sum_{u>v>0} \frac{q^{(s-1)(-u)+(t-1)v}}{[-u]_q^s[v]_q^t} = \sum_{u>v>0} \frac{q^{u+(t-1)v}}{[u]_q^s[v]_q^t} \\ &= q^{s+t}\zeta_1[s,t;1/q].\end{aligned}$$

Let

$$\zeta_-[s] := \sum_{n=1}^{\infty} \frac{q^{(s-1)(-n)}}{[-n]_q^s} = (-1)^s \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s}$$

and for convenience, put

$$\zeta_\pm[s] := \zeta[s] + \zeta_-[s] = \sum_{0\neq n\in\mathbf{Z}} \frac{q^{(s-1)n}}{[n]_q^s} = (-1)^s \sum_{0\neq n\in\mathbf{Z}} \frac{q^n}{[n]_q^s}.$$

Note that if $s-1$ is a positive integer and $n \neq 0$, then

$$\frac{q^n}{[n]_q^s} = \frac{q^n}{[n]_q^2}\left(1-q+\frac{q^n}{[n]_q}\right)^{s-2} = \sum_{k=0}^{s-2}\binom{s-2}{k}(1-q)^k \frac{q^{(s-1-k)n}}{[n]_q^{s-k}}$$

and so

$$\zeta_-[s] = (-1)^s \sum_{k=0}^{s-2} \binom{s-2}{k} (1-q)^k \zeta[s-k] \tag{6}$$

and

$$\zeta_\pm[s] = \big(1+(-1)^s\big)\zeta[s] + (-1)^s \sum_{k=1}^{s-2} \binom{s-2}{k} (1-q)^k \zeta[s-k] \tag{7}$$

are expressible in terms of values of the $q$-Riemann zeta function, i.e. (2) with $m=1$. Finally, as in [13], let

$$\varphi[s] := \sum_{n=1}^{\infty} \frac{(n-1)q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{nq^{(s-1)n}}{[n]_q^s} - \zeta[s].$$

We also employ the notation [24]

$$\binom{z}{a,\, b} := \binom{z}{a}\binom{z-a}{b} = \binom{z}{b}\binom{z-b}{a} = \binom{z}{a+b}\frac{(a+b)!}{a!\,b!}$$

for the trinomial coefficient, in which $a$, $b$ are nonnegative integers, and which reduces to $z!/a!\,b!(z-a-b)!$ if $z$ is an integer not less than $a+b$. We can now state our main result.

**Theorem 2 ($q$-analog of Euler's double zeta reduction).** *Let $s-1$ and $t-1$ be positive integers, and let $0<q<1$. Then*

$$\begin{aligned}
&(-1)^t\zeta_1[s,t] - (-1)^s\zeta_2[s,t] \\
&\quad= \sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a} \binom{a+t-1}{a,b} (1-q)^b \big(\zeta_\pm[s-a-b]\zeta[a+t] \\
&\qquad\qquad\qquad\qquad -\zeta[s+t-b] - (1-q)\zeta[s+t-b-1]\big) \\
&\quad+ \sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a} \binom{a+s-1}{a,b} (1-q)^b \big(\zeta_\pm[t-a-b]\zeta[a+s] \\
&\qquad\qquad\qquad\qquad -\zeta[s+t-b] - (1-q)\zeta[s+t-b-1]\big) \\
&\quad- \sum_{j=1}^{\min(s,t)} \binom{s+t-j-1}{s-j,t-j} (1-q)^{j-1} \big(2\zeta[s+t-j+1] - (1-q)\varphi[s+t-j]\big) \\
&\quad-\zeta_\pm[s]\zeta[t] + (-1)^s \sum_{k=0}^{s-1} \binom{s-1}{k} (1-q)^k \zeta[s+t-k].
\end{aligned}$$

**Corollary 1 (Euler's double zeta reduction).** *Let $s-1$ and $t-1$ be positive integers with opposite parity, and let $2h = \max(s,t)$. Then (3) holds.*

**Corollary 2.** *Let $s-1$ and $t-1$ be positive integers with like parity, and let $2h = \max(s,t)$. Then*

$$2\sum_{k=1}^{h}\left[\binom{s+t-2k-1}{s-1}+\binom{s+t-2k-1}{t-1}\right]\zeta(2k)\zeta(s+t-2k)$$
$$=\big(1+(-1)^s\big)\zeta(s)\zeta(t)+\left[\binom{s+t}{t}-(-1)^s\right]\zeta(s+t).$$

**Proofs.** Let $q\to 1$ in Theorem 2. With the obvious notation

$$\zeta_{\pm}(s):=\lim_{q\to 1}\zeta_{\pm}[s]=\sum_{0\neq n\in\mathbf{Z}}\frac{1}{n^s}=\big(1+(-1)^s\big)\zeta(s),$$

we find that

$$\begin{aligned}(-1)^t\zeta(s,t)-(-1)^s\zeta(s,t)&=\sum_{a=0}^{s-2}\binom{a+t-1}{a}\big(\zeta_{\pm}(s-a)\zeta(a+t)-\zeta(s+t)\big)\\&\quad+\sum_{a=0}^{t-2}\binom{a+s-1}{a}\big(\zeta_{\pm}(t-a)\zeta(a+s)-\zeta(s+t)\big)\\&\quad-2\binom{s+t-2}{s-1}\zeta(s+t)-\zeta_{\pm}(s)\zeta(t)+(-1)^s\zeta(s+t).\end{aligned}$$

Since

$$\sum_{a=0}^{s-2}\binom{a+t-1}{a}=\binom{s+t-2}{t}\quad\text{and}\quad\sum_{a=0}^{t-2}\binom{a+s-1}{a}=\binom{s+t-2}{s},$$

and

$$\binom{s+t-2}{t}+\binom{s+t-2}{s}+2\binom{s+t-2}{s-1}=\binom{s+t}{t},$$

it follows that

$$\begin{aligned}&(-1)^t\zeta(s,t)-(-1)^s\zeta(s,t)\\&=\sum_{a=0}^{s-2}\binom{a+t-1}{a}\zeta_{\pm}(s-a)\zeta(a+t)+\sum_{a=0}^{t-2}\binom{a+s-1}{a}\zeta_{\pm}(t-a)\zeta(a+s)\\&\quad-\left[\binom{s+t-2}{t}+\binom{s+t-2}{s}+2\binom{s+t-2}{s-1}-(-1)^s\right]\zeta(s+t)-\zeta_{\pm}(s)\zeta(t)\\&=\sum_{j=2}^{s}\binom{s+t-j-1}{t-1}\zeta_{\pm}(j)\zeta(s+t-j)+\sum_{j=2}^{t}\binom{s+t-j-1}{s-1}\zeta_{\pm}(j)\zeta(s+t-j)\\&\quad-\left[\binom{s+t}{t}-(-1)^s\right]\zeta(s+t)-\zeta_{\pm}(s)\zeta(t)\\&=2\sum_{k=1}^{s/2}\binom{s+t-2k-1}{t-1}\zeta(2k)\zeta(s+t-2k)+2\sum_{k=1}^{t/2}\binom{s+t-2k-1}{s-1}\zeta(2k)\zeta(s+t-2k)\end{aligned}$$

$$-\big(1+(-1)^s\big)\zeta(s)\zeta(t)-\left[\binom{s+t}{t}-(-1)^s\right]\zeta(s+t). \tag{8}$$

Since the binomial coefficients vanish if $k$ exceeds the indicated range of summation above, we can replace the two sums by a single sum on $k$ ranging from 1 up to $h$. If $s$ and $t$ have opposite parity, multiply both sides by $(-1)^t=(-1)^{s+1}$ and divide each term by 2 to complete the proof of Corollary 1. For Corollary 2, note that if $s$ and $t$ have like parity, then the left hand side of (8) vanishes. □

## 3 Proof of Theorem 2

The key ingredient is the following partial fraction decomposition.

**Lemma 1 (cf. Lemma 3.1 of [13] and Lemma 1 of [24]).** *If $s$ and $t$ are positive integers, and $u$ and $v$ are non-zero real numbers such that $u+v\neq 0$, then*

$$\begin{aligned}\frac{1}{[u]_q^s[v]_q^t} &= \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{a,b}\frac{(1-q)^b q^{(t-1-b)u+av}}{[u]_q^{s-a-b}[u+v]_q^{a+t}}\\ &\quad+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{a,b}\frac{(1-q)^b q^{au+(s-1-b)v}}{[v]_q^{t-a-b}[u+v]_q^{a+s}}\\ &\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^j q^{(t-j)u+(s-j)v}}{[u+v]_q^{s+t-j}}.\end{aligned}$$

*Proof.* As in [13], let $x$ and $y$ be non-zero real numbers such that $x+y+(q-1)xy\neq 0$. Apply the partial differential operator

$$\frac{1}{(r-1)!}\left(-\frac{\partial}{\partial x}\right)^{r-1}\frac{1}{(s-1)!}\left(-\frac{\partial}{\partial y}\right)^{s-1}$$

to both sides of the identity

$$\frac{1}{xy}=\frac{1}{x+y+(q-1)xy}\left(\frac{1}{x}+\frac{1}{y}+q-1\right);$$

then let $x=[u]_q$, $y=[v]_q$ and observe that then $x+y+(q-1)xy=[u+v]_q$. □

We now proceed with the proof of Theorem 2. First, multiply both sides of Lemma 1 by $q^{(s-1)u+(t-1)v}$ to obtain

$$\begin{aligned}\frac{q^{(s-1)u}q^{(t-1)v}}{[u]_q^s[v]_q^t} &= \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{a,b}\frac{(1-q)^b q^{(s-a-b-1)u}q^{(a+t-1)(u+v)}}{[u]_q^{s-a-b}[u+v]_q^{a+t}}\\ &\quad+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{a,b}\frac{(1-q)^b q^{(t-a-b-1)v}q^{(a+s-1)(u+v)}}{[v]_q^{t-a-b}[u+v]_q^{a+s}}\end{aligned}$$

$$-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^j\,q^{(s+t-j-1)(u+v)}}{[u+v]_q^{s+t-j}}.$$

After replacing $u$ by $u-v$ and $v$ by $-v$, we find that

$$\begin{aligned}\frac{q^{(s-1)(u+v)}q^{(t-1)(-v)}}{[u+v]_q^s[-v]_q^t} &= \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{a,b}\frac{(1-q)^b q^{(s-a-b-1)(u+v)}q^{(a+t-1)u}}{[u+v]_q^{s-a-b}[u]_q^{a+t}}\\ &\quad+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{a,b}\frac{(1-q)^b q^{(t-a-b-1)(-v)}q^{(a+s-1)u}}{[-v]_q^{t-a-b}[u]_q^{a+s}}\\ &\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^j q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}.\end{aligned} \tag{9}$$

We'd like to sum (9) over all ordered pairs of positive integers $(u,v)$, but we must exercise some care in doing so since some of the terms on the right hand side may diverge. The difficulty can be circumvented by judiciously combining the troublesome terms before summing. To this end, observe that

$$\begin{aligned}&\sum_{a=0}^{s-1}\binom{a+t-1}{a,s-1-a}\frac{(1-q)^{s-1-a}q^{(a+t-1)u}}{[u+v]_q[u]_q^{a+t}}\\ &\quad+\sum_{a=0}^{t-1}\binom{a+s-1}{a,t-1-a}\frac{(1-q)^{t-1-a}q^{(a+s-1)u}}{[-v]_q[u]_q^{a+s}}\\ &\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^j q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}\\ &=\sum_{j=1}^{s}\binom{s+t-j-1}{s-j,j-1}\frac{(1-q)^{j-1}q^{(s+t-j-1)u}}{[u+v]_q[u]_q^{s+t-j}}\\ &\quad+\sum_{j=1}^{t}\binom{s+t-j-1}{t-j,j-1}\frac{(1-q)^{j-1}q^{(s+t-j-1)u}}{[-v]_q[u]_q^{s+t-j}}\\ &\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^j q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}\\ &=\left(\frac{1}{[u+v]_q}+\frac{1}{[-v]_q}-(1-q)\right)\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^{j-1}q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}\\ &=\left(\frac{1}{[u+v]_q}-\frac{1}{[v]_q}\right)\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^{j-1}q^{(s+t-j-1)u}}{[u]_q^{s+t-j}},\end{aligned} \tag{10}$$

where we have used the fact that

$$\binom{s+t-j-1}{s-j,j-1}=\binom{s+t-j-1}{t-j,j-1}=\binom{s+t-j-1}{s-j,t-j}$$

vanishes if $j > \min(s,t)$. Substituting (10) into (9) yields

$$\begin{aligned}
&\frac{q^{(s-1)(u+v)}q^{(t-1)(-v)}}{[u+v]_q^s[-v]_q^t}\\
&=\sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}\frac{(1-q)^b q^{(s-a-b-1)(u+v)}q^{(a+t-1)u}}{[u+v]_q^{s-a-b}[u]_q^{a+t}}\\
&+\sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}\frac{(1-q)^b q^{(t-a-b-1)(-v)}q^{(a+s-1)u}}{[-v]_q^{t-a-b}[u]_q^{a+s}}\\
&-\left(\frac{1}{[v]_q}-\frac{1}{[u+v]_q}\right)\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}\frac{(1-q)^{j-1}q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}. \qquad (11)
\end{aligned}$$

Now assume that $s > 1$. Then

$$\sum_{u,v=1}^{\infty}\left(\frac{1}{[v]_q}-\frac{1}{[u+v]_q}\right)\frac{q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}=\sum_{n=1}^{\infty}\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j}}\sum_{k>n}\left(\frac{1}{[k-n]_q}-\frac{1}{[k]_q}\right).$$

Recalling that $0 < q < 1$, we evaluate the telescoping sum

$$\begin{aligned}
\sum_{k>n}\left(\frac{1}{[k-n]_q}-\frac{1}{[k]_q}\right) &= \lim_{N\to\infty}\sum_{k=n+1}^{n+N}\left(\frac{1}{[k-n]_q}-\frac{1}{[k]_q}\right)\\
&= \lim_{N\to\infty}\sum_{k=1}^{n}\left(\frac{1}{[k]_q}-\frac{1}{[N+k]_q}\right)\\
&= (q-1)n+\sum_{k=1}^{n}\frac{1}{[k]_q},
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{u,v=1}^{\infty}\left(\frac{1}{[v]_q}-\frac{1}{[u+v]_q}\right)\frac{q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}\\
&\qquad=(q-1)\big(\varphi[s+t-j]+\zeta[s+t-j]\big)+\sum_{n=1}^{\infty}\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j}}\sum_{k=1}^{n}\frac{1}{[k]_q}.
\end{aligned}$$

But this last double sum evaluates as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j}}\sum_{k=1}^{n}\frac{1}{[k]_q}\\
&=\sum_{n>k>0}\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j}[k]_q}+\sum_{n=1}^{\infty}\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j+1}}\\
&=\zeta[s+t-j,1]+\sum_{n=1}^{\infty}\frac{q^{(s+t-j)n}}{[n]_q^{s+t-j+1}}+\sum_{n=1}^{\infty}\frac{q^{(s+t-j-1)n}-q^{(s+t-j)n}}{[n]_q^{s+t-j+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \zeta[s+t-j,1]+\zeta[s+t-j+1]+(1-q)\sum_{n=1}^{\infty}\left(\frac{1-q^n}{1-q}\right)\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j+1}}\\
&= \zeta[s+t-j,1]+\zeta[s+t-j+1]+(1-q)\sum_{n=1}^{\infty}\frac{q^{(s+t-j-1)n}}{[n]_q^{s+t-j}}\\
&= \zeta[s+t-j,1]+\zeta[s+t-j+1]+(1-q)\zeta[s+t-j].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{u,v=1}^{\infty}\left(\frac{1}{[v]_q}-\frac{1}{[u+v]_q}\right)\frac{q^{(s+t-j-1)u}}{[u]_q^{s+t-j}}\\
&\qquad = \zeta[s+t-j,1]+\zeta[s+t-j+1]+(q-1)\varphi[s+t-j].
\end{aligned}
$$

Consequently, summing (11) over all ordered pairs of positive integers $(u,v)$ yields

$$
\begin{aligned}
(-1)^t\zeta_1[s,t] &= \sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}(1-q)^b\zeta[s-a-b,a+t]\\
&\quad+\sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}(1-q)^b\zeta_-[t-a-b]\zeta[a+s]\\
&\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^{j-1}\\
&\qquad\times\big(\zeta[s+t-j,1]+\zeta[s+t-j+1]-(1-q)\varphi[s+t-j]\big). \quad (12)
\end{aligned}
$$

Now assume also that $t>1$. For each pair of integers $(a,b)$ with $0\le a\le s-1$, $0\le b\le s-2-a$, we apply the $q$-stuffle multiplication rule [11, eq. (2.2)] in the form

$$
\begin{aligned}
\zeta[s-a-b]\zeta[a+t] &= \zeta[s-a-b,a+t]+\zeta[a+t,s-a-b]\\
&\qquad+\zeta[s+t-b]+(1-q)\zeta[s+t-b-1],
\end{aligned}
$$

substituting for $\zeta[s-a-b,a+t]$ in (12). Thus, we find that

$$
\begin{aligned}
(-1)^t\zeta_1[s,t] &= \sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}(1-q)^b\big(\zeta[s-a-b]\zeta[a+t]-\zeta[s+t-b]\\
&\qquad -(1-q)\zeta[s+t-b-1]-\zeta[a+t,s-a-b]\big)\\
&\quad+\sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}(1-q)^b\zeta_-[t-a-b]\zeta[a+s]\\
&\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^{j-1}\\
&\qquad\times\big(\zeta[s+t-j,1]+\zeta[s+t-j+1]-(1-q)\varphi[s+t-j]\big).
\end{aligned}
$$

The sum of $\zeta[s+t-j,1]$ over $j$ can be combined with the double sum of $\zeta[a+t,s-a-b]$ over $a$ and $b$ by extending the range of the latter to include the value $b=s-1-a$. Doing this yields

$$\begin{aligned}(-1)^t\zeta_1[s,t] &= \sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}(1-q)^b\big(\zeta[s-a-b]\zeta[a+t]\\ &\qquad -\zeta[s+t-b]-(1-q)\zeta[s+t-b-1]\big)\\ &\quad+\sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}(1-q)^b\zeta_-[t-a-b]\zeta[a+s]\\ &\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^{j-1}\\ &\qquad\times\big(\zeta[s+t-j+1]-(1-q)\varphi[s+t-j]\big)\\ &\quad-\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{a,b}(1-q)^b\zeta[t+a,s-a-b].\end{aligned}$$

It follows that for integers $s>1$ and $t>1$,

$$\begin{aligned}&(-1)^s\zeta_1[t,s]+(-1)^t\zeta_1[s,t]\\ &=\sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}(1-q)^b\big(\zeta_\pm[s-a-b]\zeta[a+t]\\ &\qquad -\zeta[s+t-b]-(1-q)\zeta[s+t-b-1]\big)\\ &+\sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}(1-q)^b\big(\zeta_\pm[t-a-b]\zeta[a+s]\\ &\qquad -\zeta[s+t-b]-(1-q)\zeta[s+t-b-1]\big)\\ &-2\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^{j-1}\big(\zeta[s+t-j+1]-(1-q)\varphi[s+t-j]\big)\\ &-\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{a,b}(1-q)^b\zeta[t+a,s-a-b]\\ &-\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{a,b}(1-q)^b\zeta[s+a,t-a-b].\end{aligned}\tag{13}$$

By Theorem 2.1 of [13],

$$\begin{aligned}\zeta[s]\zeta[t] &= \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{a,b}(1-q)^b\zeta[t+a,s-a-b]\\ &\quad+\sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{a,b}(1-q)^b\zeta[s+a,t-a-b]\end{aligned}$$

$$-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^j\varphi[s+t-j].$$

We use this latter decomposition formula to eliminate the last two sums of double $q$-zeta values in (13), obtaining

$$\begin{aligned}
&(-1)^s\zeta_1[t,s]+(-1)^t\zeta_1[s,t]+\zeta[s]\zeta[t]\\
&\quad=\sum_{a=0}^{s-2}\sum_{b=0}^{s-2-a}\binom{a+t-1}{a,b}(1-q)^b\big(\zeta_\pm[s-a-b]\zeta[a+t]\\
&\qquad\qquad\qquad\qquad-\zeta[s+t-b]-(1-q)\zeta[s+t-b-1]\big)\\
&\quad+\sum_{a=0}^{t-2}\sum_{b=0}^{t-2-a}\binom{a+s-1}{a,b}(1-q)^b\big(\zeta_\pm[t-a-b]\zeta[a+s]\\
&\qquad\qquad\qquad\qquad-\zeta[s+t-b]-(1-q)\zeta[s+t-b-1]\big)\\
&\quad-\sum_{j=1}^{\min(s,t)}\binom{s+t-j-1}{s-j,t-j}(1-q)^{j-1}\\
&\qquad\times\big(2\zeta[s+t-j+1]-(1-q)\varphi[s+t-j]\big).
\end{aligned}\tag{14}$$

But

$$\begin{aligned}
\zeta_-[s]\zeta[t]&=(-1)^s\sum_{u=1}^{\infty}\frac{q^u}{[u]_q^s}\sum_{v=1}^{\infty}\frac{q^{(t-1)v}}{[v]_q^t}\\
&=(-1)^s\sum_{u>v>0}\frac{q^{u+(t-1)v}}{[u]_q^s[v]_q^t}+(-1)^s\sum_{v>u>0}\frac{q^{(t-1)v+u}}{[v]_q^t[u]_q^s}+(-1)^s\sum_{v=1}^{\infty}\frac{q^{tv}}{[v]_q^{s+t}}.
\end{aligned}$$

Since

$$\frac{q^{tv}}{[v]_q^{s+t}}=\frac{q^{tv}}{[v]_q^{t+1}}\left(1-q+\frac{q^v}{[v]_q}\right)^{s-1}=\sum_{k=0}^{s-1}\binom{s-1}{k}\frac{(1-q)^kq^{(s+t-k-1)v}}{[v]_q^{s+t-k}},$$

it follows that

$$\sum_{v=1}^{\infty}\frac{q^{tv}}{[v]_q^{s+t}}=\sum_{k=0}^{s-1}\binom{s-1}{k}(1-q)^k\zeta[s+t-k],$$

and therefore

$$\zeta_-[s]\zeta[t]=(-1)^s\zeta_2[s,t]+(-1)^s\zeta_1[t,s]+(-1)^s\sum_{k=0}^{s-1}\binom{s-1}{k}(1-q)^k\zeta[s+t-k].$$

We now use this formula to substitute the initial $(-1)^s\zeta_1[t,s]$ term in (14) to complete the proof. □

## References


1. Berndt, B.: Ramanujan's Notebooks Part I. Springer, New York (1985)
2. Borwein, D., Borwein, J.M., Bradley, D.M.: Parametric Euler sum identities. J. Math. Anal. Appl., **316**, no. 1, 328–338 (2006) doi: 10.1016/j.jmaa.2005.04.040
3. Borwein, J.M., Bradley, D.M., Broadhurst, D.J.: Evaluations of $k$-fold Euler/Zagier sums: a compendium of results for arbitrary $k$. Electron. J. Combin., **4**, no. 2, #R5 (1997) Wilf Festschrift
4. Borwein, J.M., Bradley, D.M., Broadhurst, D.J., Lisoněk, P.: Combinatorial aspects of multiple zeta values. Electron. J. Combin., **5**, no. 1, #R38 (1998)
5. Borwein, J.M., Bradley, D.M., Broadhurst, D.J., and Lisoněk, P.: Special values of multiple polylogarithms. Trans. Amer. Math. Soc., **353**, no. 3, 907–941 (2001).
6. Bowman, D., and Bradley, D.M.: Multiple polylogarithms: a brief survey. In: Berndt, B.C., Ono, K. (eds.) Proceedings of a Conference on $q$-Series with Applications to Combinatorics, Number Theory and Physics, pp. 71–92. Contemporary Math., **291**, Amer. Math. Soc., Providence (2001)
7. Bowman, D., and Bradley, D.M.: The algebra and combinatorics of shuffles and multiple zeta values. J. Combin. Theory, Ser. A, **97**, no. 1, 43–61 (2002)
8. Bowman, D., and Bradley, D.M.: Resolution of some open problems concerning multiple zeta evaluations of arbitrary depth. Compositio Math., **139**, no. 1, 85–100 (2003) doi: 10.1023/B:COMP:0000005036.52387.da
9. Bowman, D., Bradley, D.M., Ryoo, J.: Some multi-set inclusions associated with shuffle convolutions and multiple zeta values, European J. Combin., **24**, no. 1, 121–127 (2003)
10. Bradley, D.M.: Partition identities for the multiple zeta function. In: Aoki, T., Kanemitsu, S., Nakahara, M., Ohno, Y. (eds.) Zeta Functions, Topology, and Quantum Physics, pp. 19–29. Developments in Mathematics, **14**, Springer-Verlag, New York (2005)
11. Bradley, D.M.: Multiple $q$-zeta values. J. Algebra, **283**, no. 2, 752–798 (2005) doi: 10.1016/j.jalgebra.2004.09.017
12. Bradley, D.M.: Duality for finite multiple harmonic $q$-series. Discrete Math., **300**, no. 1–3, 44–56 (2005) doi: 10.1016/j.disc.2005.06.008 [MR 2170113] (2006m:05019)
13. Bradley, D.M.: A $q$-analog of Euler's decomposition formula for the double zeta function. Internat. J. Math. Math. Sci., **2005**, no. 21, 3453–3458 (2005) doi:10.1155/IJMMS.2005.3453 [MR 2206867] (2006k:11174)
14. Bradley, D.M.: On the sum formula for multiple $q$-zeta values. Rocky Mountain J. Math., **37**, no. 5, 1427–1434 (2007).
15. Broadhurst, D.J., and Kreimer, D.: Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. Phys. Lett. B, **393**, no. 3-4, 403–412 (1997)
16. Cartier, P.: Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents. Astérisque, **282**, viii, 137–173 (2002)
17. Euler, L: Meditationes Circa Singulare Serierum Genus, Novi Comm. Acad. Sci. Petropol., **20**, 140–186 (1775). Reprinted in Opera Omnia, ser. I, **15**, 217–267. B. G. Teubner, Berlin (1927)
18. Euler, L.: Briefwechsel. **1**, Birhäuser, Basel, (1975)
19. Euler, L., Goldbach, C.: Briefwechsel. 1729–1764, Akademie-Verlag, Berlin (1965)
20. Le, T.Q.T., and Murakami, J.: Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions. Topology Appl., **62**, no. 2, 193–206 (1995)
21. Okuda, J., and Takeyama, Y.: On relations for the multiple $q$-zeta values.Ramanujan J., **14**, no. 3, 379–387 (2007)
22. Waldschmidt, M.: Valeurs zêta multiples: une introduction. J. Théor. Nombres Bordeaux, **12**, no. 2, 581–595 (2000)
23. Zhao, J.: Multiple $q$-zeta functions and multiple $q$-polylogarithms,. Ramanujan J., **14**, no. 2, 189–221 (2007)
24. Zhou, X., Cai, T., Bradley, D.M.: Signed $q$-analogs of Tornheim's double series. Proc. Amer. Math. Soc., **136**, no. 8, 2689–2698 (2008)

25. Zudilin, V.V.: Algebraic relations for multiple zeta values. (Russian), Uspekhi Mat. Nauk, **58**, no. 1, 3–32 (2003). Translation in Russian Math. Surveys, **58**, no. 1, 1–29 (2003)